A Few Attempts at Visualizing Shimura Varieties

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Outline

What's a Shimura Variety?

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- **Langlands enthusiast:** all of the above!

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Example:

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where $Y(N) := \Gamma(N)\backslash\mathcal{H}$ is the modular curve and $\Gamma(N)$ is the principal congruence subgroup of level N.

Modular Curve $Y(1)$

Let's look at the modular curve $Y(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$.

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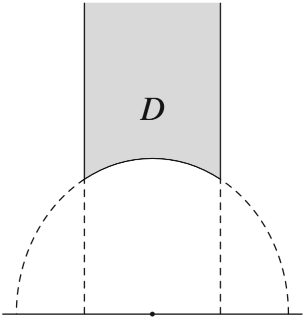
It has the following fundamental domain [DS05]:

$$D = \left\{\tau \in \mathcal{H} : |\mathrm{Re}(\tau)| \leq \frac{1}{2}, |\tau| \geq 1\right\}$$



Modular Curve $Y(1)$ as a Moduli Space

Each point $\tau \in \mathcal{H}$ gives rise to a complex elliptic curve:

$$E_\tau = \mathbb{C}/\Lambda_\tau \qquad \text{where } \Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$$

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For two points $\tau_1, \tau_2 \in \mathcal{H}$,

$$\mathrm{SL}_2(\mathbb{Z})\tau_1 = \mathrm{SL}_2(\mathbb{Z})\tau_2 \iff \Lambda_{\tau_1} \sim \Lambda_{\tau_2} \iff \mathbb{C}/\Lambda_{\tau_1} \cong \mathbb{C}/\Lambda_{\tau_2}$$

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Thus, points in the fundamental domain D correspond to unique isomorphism classes of elliptic curves.

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Problem: which torus do we choose for each elliptic curve?

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Then the preimage $\eta^{-1}(C)$ via the Hopf fibration $\eta : \mathbb{S}^3 \to \mathbb{S}^2$ is isometric to $\mathbb{C}/\Lambda$ for $\Lambda = 2\pi\mathbb{Z} \oplus \left(\frac{A}{2} + i\frac{L}{2}\right)\mathbb{Z}$.

Visualizing the Modular Curve $Y(1)$ as a Moduli Space

Thus, I want a continuous map

$$\tau \in D \longrightarrow \text{curve } C_\tau \text{ s.t. } 2\pi\mathbb{Z} \oplus \left(\frac{A_{C_\tau}}{2} + i\frac{L_{C_\tau}}{2}\right)\mathbb{Z} \sim \Lambda_\tau$$

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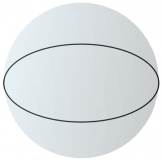
Actually, it makes (some) sense to change the fundamental domain: the equator of the sphere is a natural starting point and has $A = 2\pi, L = 2\pi$



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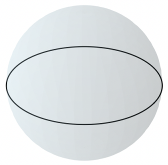

The corresponding lattice is

$$\Lambda = 2\pi\mathbb{Z} \oplus (\pi + i\pi)\mathbb{Z} \sim \mathbb{Z} \oplus \frac{1+i}{2}\mathbb{Z}$$

So we should pick a fundamental domain which contains $\frac{1+i}{2}$.

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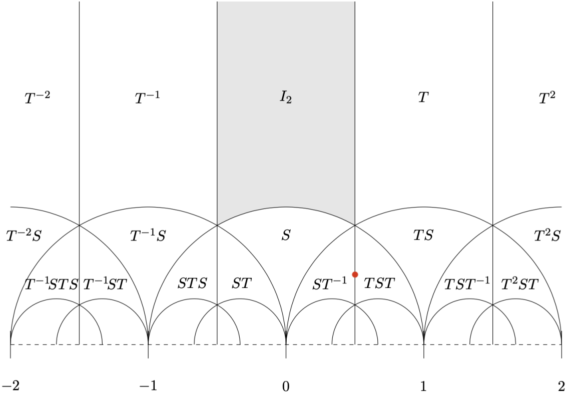
The figure below ([Connd]) shows a few of the transforms of D by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which generate $\mathrm{SL}_2(\mathbb{Z})$:



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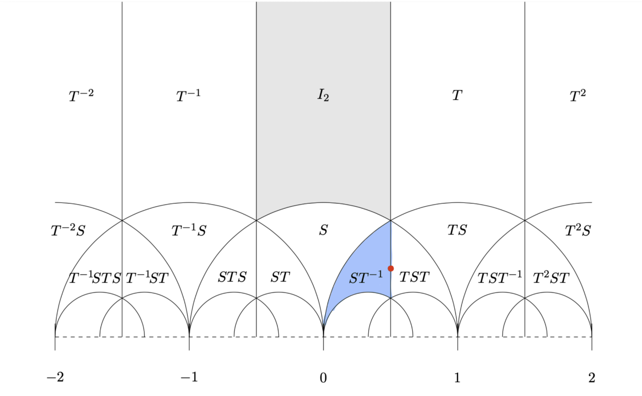
$ST^{-1}(D)$ is the only transform containing $\frac{1+i}{2}$ for which curves on the sphere can be constructed.



Caveat: not every point in $ST^{-1}(D)$ can be reached.

Choice of Curves

I chose curves of the following form:

$$\phi(t) = \frac{2\pi}{\phi_0} \int_0^t \cos(A \sin(nu))\, du$$

$$\theta(t) = \frac{\pi}{2} - B\Big(\int_0^t \sin(A \sin(nu))\, du - \theta_0 \Big) - k$$

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The parameters are:

- A: curviness/bulbiness
- B: amplitude
- k: vertical offset
- n: number of bulbs

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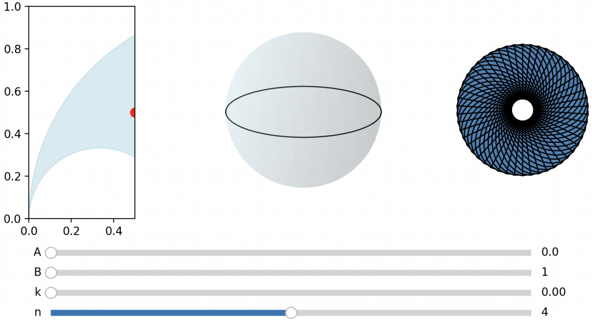
Example 1



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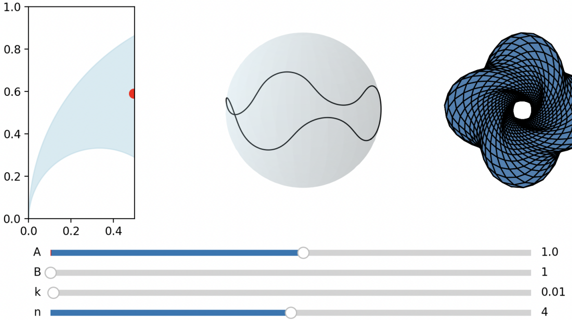

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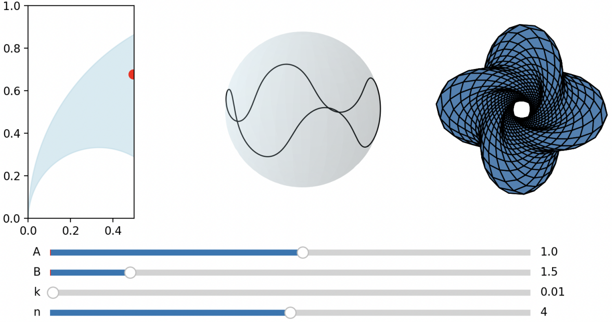

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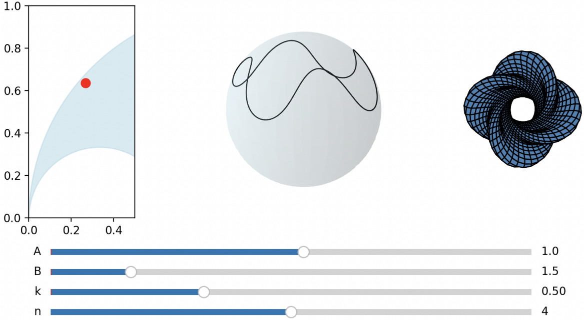
Example 4



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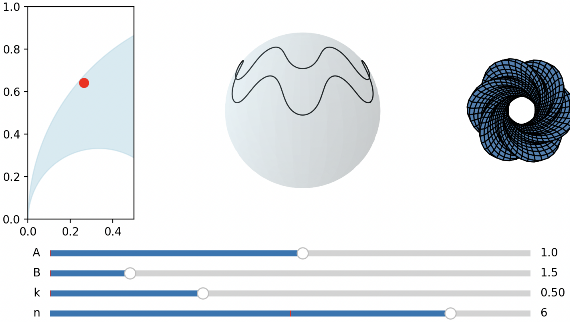
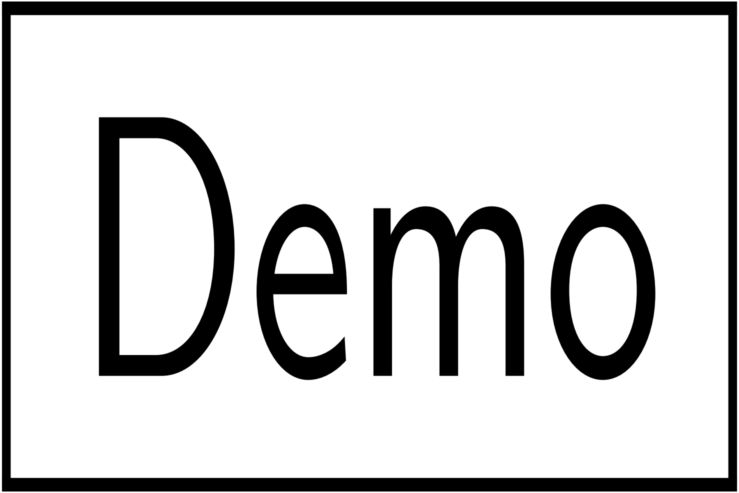
Example 5



Video Demo



Future Work

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- Make visualization work with standard fundamental domain

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- Choose exactly one curve per $\tau \in D$
- Allow for level structures, e.g. highlight an order N subgroup

Ekedahl-Oort Types

An elliptic curve E over $\mathbb{F}_p$ comes in two flavors:

- **Ordinary** if $E[p] \cong L := \mathbb{Z}/p\mathbb{Z} \oplus \mu_p$
- **Supersingular** if $E[p] \cong I_{1,1}$ where $0 \to \alpha_p \to I_{1,1} \to \alpha_p \to 0$ is a non-split exact sequence

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A higher dimensional (principally polarized) abelian variety A over $\mathbb{F}_p$ can be categorized into *Ekedahl-Oort types* based on the group scheme $A[p]$.

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A higher dimensional (principally polarized) abelian variety A over $\mathbb{F}_p$ can be categorized into *Ekedahl-Oort types* based on the group scheme $A[p]$.

Example: 2-dimensional abelian varieties come in 4 Ekedahl-Oort types (see [Pri06]):

- $A[p] \cong L^2$ *(ordinary)*
- $A[p] \cong L \oplus I_{1,1}$
- $A[p] \cong I_{2,1}$
- $A[p] \cong I_{1,1}^2$ *(superspecial)*

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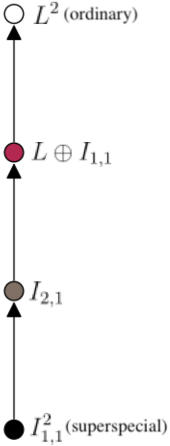
Goal: construct a diagram to visualize the structure of the Ekedahl-Oort stratification.

Visualizing the Ekedahl-Oort Stratification

Example: $G = \mathrm{GSp}_4$ (Siegel modular variety) i.e. $\mathcal{A}_{2,\mathbb{F}_p}$



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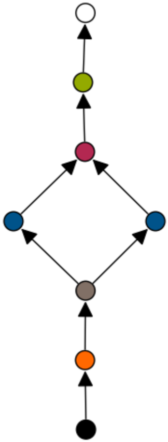
Example: $G = \mathrm{GSp}_6$ (Siegel modular variety) i.e. $\mathcal{A}_{3,\mathbb{F}_p}$



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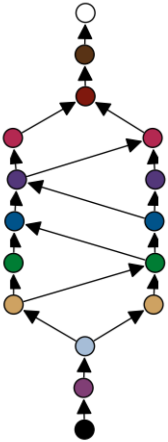
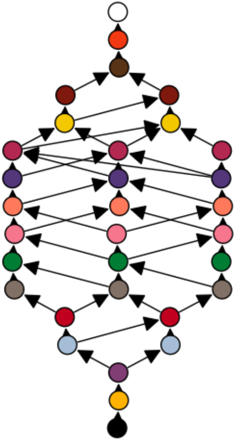
Example: $G = \mathrm{GSp}_8$ (Siegel modular variety) i.e. $\mathcal{A}_{4,\mathbb{F}_p}$



Visualizing the Ekedahl-Oort Stratification

Example: $G = \mathrm{GSp}_{10}$ (Siegel modular variety) i.e. $\mathcal{A}_{5,\mathbb{F}_p}$



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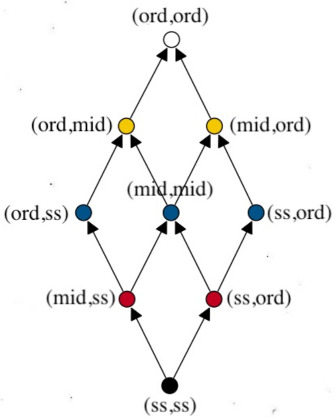
Example: $G = \mathrm{SU}(2,1)$; this gives a moduli space for 3-dimensional abelian varieties with a "signature $(2,1)$ condition".

Visualizing the Ekedahl-Oort Stratification

Example: $G = \mathrm{SU}(2,1) \times \mathrm{SU}(2,1)$; this gives a moduli space for pairs of 3-dimensional abelian varieties (A_1, A_2) with a "signature $(2,1)$ condition".



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- Handle cases with non-trivial Frobenius action on root system

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- Connect visualization to Bruhat-Tits building

Thank you!

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