Strata Hasse Invariants on Proper Abelian Type Shimura Varieties and Applications to Galois Representations

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Abstract

We partially extend the results of [GK19a] by constructing Hasse invariants on good reductions of proper abelian type Shimura varieties, and using them to associate Galois representations to systems of Hecke eigenvalues. This is made possible by (1) using the Griffiths bundle, as described by Goldring in [Gol19], as a generalization of the Hodge line bundle, and (2) using the abelian type Ekedahl-Oort stratification constructed by Shen and Zhang in [SZ22] and independently by Imai, Kato, and Youcis in [IKY23]. Our main contribution is utilizing these constructions together under the main ideas of [GK19a].

1 Introduction

The goal of this paper is to apply the theory of *strata Hasse invariants*, as described by Goldring and Koskivirta in [GK19a], to the setting of proper abelian type Shimura varieties. In doing so, we show that the applications to the Langlands correspondence in [GK19a] extend to the proper abelian type case. We discuss the main ideas here.

1.1 Strata Hasse Invariants

1.1.1 Hasse Invariants and the Ekedahl-Oort Stratification

The classical *Hasse invariant* is a binary invariant for abelian varieties in characteristic $p > 0$ which distinguishes ordinary abelian varieties (those with maximal p-rank) from non-ordinary ones. From another viewpoint, the Hasse invariant is a "mod p modular form" which distinguishes the *ordinary locus* of the moduli space of abelian varieties $\mathcal{A}_{g,\mathbb{F}_p}$ in characteristic p from the non-ordinary locus. In the case of elliptic curves, this is the end of the story; an elliptic curve in characteristic p is either ordinary or supersingular. However, higher dimensional abelian varieties come in several flavors of "non-ordinariness." Additionally, a general Hodge type Shimura variety may have no ordinary locus; the classical Hasse invariant is precisely 0 on such a variety. Thus, a finer invariant is needed.

The *Ekedahl-Oort stratification* is one way to stratify $\mathcal{A}_{g,\mathbb{F}_p}$ into various levels of "ordinariness." Its history is described in §3.2.1. Each stratum is determined by discrete data in the form of an (isomorphism class of an) *F-zip*. In other words, the EO-stratification can be given by a morphism of stacks

$$\zeta : \mathcal{A}_{g,\mathbb{F}_p} \to F\text{-}\mathbf{Zip}^{\tau}$$

where F-$\mathbf{Zip}^{\tau}$ is the stack of F-zips of type τ. The work of Pink-Wedhorn-Ziegler ([PWZ11]) showed that the EO-stratification of a general Shimura variety can be given in the form of a morphism of stacks

$$\zeta : \mathcal{S}_K(G, X) \to G\text{-}\mathbf{Zip}^{\mu}$$

where $\mathcal{S}_K(G, X)$ is the special fiber of the integral model of a Shimura variety, and G-$\mathbf{Zip}^{\mu}$ is the stack of *G-zips* of type μ. The underlying topological space of this stack is in bijection with a quotient ${}^I W$ of the Weyl group of G. For each $w \in {}^I W$, denote the corresponding open stratum by $\mathcal{S}_w$ and its closure by $\overline{\mathcal{S}}_w$.

Thus, a natural generalization of the classical Hasse invariant is a collection of "mod p automorphic forms" on a good reduction of a Shimura variety which distinguish the Ekedahl-Oort strata. In other words, we want an automorphic line bundle ω on a mod p Shimura variety $\mathcal{S}_K$ such that for each $w \in {}^I W$, there exists a section $h_w \in H^0(\overline{\mathcal{S}}_w, \omega^N)$ for some N such that the non-vanishing locus of h_w is $\mathcal{S}_w$.

In the Hodge type case, the morphism ζ was constructed by Zhang in [Zha18] and the strata Hasse invariants were constructed by Goldring and Koskivirta in [GK19a] as sections of the classical "Hodge line bundle."

1.1.2 Recent Results on Abelian Type Shimura Varieties

The construction of strata Hasse invariants on a general Shimura variety requires two main ingredients:

(i) The EO stratification map $\zeta : \mathcal{S}_K \to G\text{-}\mathbf{Zip}^{\mu}$

(ii) An automorphic line bundle ω which exhibits "group-theoretic Hasse invariants" on the stack of G-zips

Given these ingredients, the group-theoretic Hasse invarints h_w on G-$\mathbf{Zip}^{\mu}$ can be pulled back via ζ to produce strata Hasse invariants on $\mathcal{S}_K$. As discussed above, Zhang [Zha18] provided the map ζ in the Hodge type case. Meanwhile, Goldring and Koskivirta gave a general framework in [GK19a] for showing that an automorphic line bundle exhibits group-theoretic Hasse invariants.

These ingredients have recently been provided in the abelian type case. The morphism ζ was constructed by Shen and Zhang in [SZ22]; a more refined construction using prismatic cohomology was given by Imai-Kato-Youcis in [IKY23]. Goldring ([Gol19]) showed that the *Griffiths bundle* (see §2.3.2) admits group-theoretic Hasse invariants on G-$\mathbf{Zip}^{\mu}$; in fact, when (G, X) is a Hodge type Shimura datum, the Griffiths bundle is precisely the Hodge bundle.

1.2 Applications to Galois Representations

We describe the main applications to Galois representations. Given the construction of strata Hasse invariants, Galois representations can be associated to automorphic representations as in [GK19a]. This follows the strategy initiated by Deligne and Serre in [DS74], in which they use a characteristic 0 lift of the classical Hasse invariant in order to provide congruences between modular forms, and in turn attach Galois representations to weight 1 (i.e. non-cohomological) modular forms.

1.2.1 Factorization

The main ingredient needed in the application to Galois representations is the following factorization theorem, which was stated as Theorem I.3.1 in [GK19a]. Let $\mathscr{S}_K$ denote an integral model of an abelian type Shimura variety $\mathrm{Sh}_K(G, X)$, and let $\mathcal{S}_K^n$ denote the reduction mod p^n. Write $\mathcal{S}_K := \mathcal{S}_K^1$. For a character η, write $\mathcal{V}(\eta)$ for the corresponding automorphic vector bundle on $\mathscr{S}_K$ (see §3.1.3). Write $\mathcal{H}$ for the global, unramified, prime-to-p Hecke algebra of G and $\mathcal{H}^{i,n}(\eta)$ its image in $\mathrm{End}(H^i(\mathcal{S}_K^n, \mathcal{V}(\eta)))$.

Theorem 1.1. (Theorem 4.4) *Let (G, X) be a proper abelian type Shimura datum. Let $\mathcal{V}(\eta)$ be an automorphic representation on $\mathscr{S}_K(G, X)$ and $i \geq 0$ an integer. For every $\delta \in \mathbb{R}_{\geq 0}$ there exists a δ-regular (Definition 2.17) weight η' such that*

$$\mathcal{H} \twoheadrightarrow \mathcal{H}^{i,n}(\eta) \quad \textit{factors through} \quad \mathcal{H} \twoheadrightarrow \mathcal{H}^{0,n}(\eta')$$

1.2.2 Galois Representations

Keep the notation from the previous section. Denote by ${}^L G$ the Langlands dual group of G. Fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. For an automorphic representation π of G, write

$$R_{p,\iota}(\pi) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to {}^L G(\overline{\mathbb{Q}}_p)$$

for the Galois representation associated to π, as predicted by the Langlands correspondence (see §4.1.2 for more details). The main results on Galois representations are summarized by the following theorem, which is stated as Theorem I.3.2 in [GK19a].

Theorem 1.2. (Theorem 4.9 and Theorem 4.10) *Let (G, X) be a proper abelian type Shimura datum and fix $\delta \in \mathbb{R}_{\geq 0}$. Assume that $r : {}^L G \to \mathrm{GL}_m$ is a representation such that $r \circ R_{p,\iota}(\pi')$ exists for all C-algebraic δ-regular π'. Then*

1. *For every triple (i, n, η) there exists a continuous Galois pseudo-representation*

$$R_{p,\iota}(r; i, n, \eta) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathcal{H}^{i,n}(\eta)$$

   which maps Frobenius to Hecke operators (see §4.1.5).

2. *If π is an automorphic representation of G with π_∞ a non-degenerate, C-algebraic limit of discrete series and π_p unramified, then $r \circ R_{p,\iota}(\pi)$ exists.*

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2 Group Theoretic Hasse Invariants

This section summarizes the main results of [GK19a] and [GK19b] in the construction of group theoretic Hasse invariants on the stack of G-zips. We fix a prime p and an algebraic closure k of $\mathbb{F}_p$.

2.1 Zip Data and the Stack of G-Zips

2.1.1 Zip Data

For our purposes, a *zip datum* is a tuple $\mathcal{Z} = (G, P, L, Q, M, \varphi)$ where G is a reductive group over $\mathbb{F}_p$, $\varphi : G \to G$ is the relative Frobenius, and $P, Q \subset G_k$ are parabolics with Levi subgroups $L \subset P$, $M \subset Q$ such that $M = \varphi(L)$. In [GK19b], these are known as *zip data of exponent 1*. A morphism between two zip data $\mathcal{Z}_1, \mathcal{Z}_2$ is a group homomorphism $f : G_1 \to G_2$ (defined over $\mathbb{F}_p$) such that

$$f(\Box_1) \subset \Box_2 \qquad \text{for } \Box = P, L, Q, M, U, V$$

where $U_i \subset P_i$ and $V_i \subset Q_i$ are the unipotent radicals. Thus, zip data form a category, which we will denote by **Zip**.

2.1.2 Cocharacter Data

Our primary source of zip data come from *cocharacter data*. Let K be a field with fixed algebraic closure $\overline{K}$. A *cocharacter datum* is a pair (G, μ) where G is a reductive group over K and $\mu : \mathbb{G}_{m,\overline{K}} \to G_{\overline{K}}$ is a cocharacter. A morphism of cocharacter data $(G_1, \mu_1) \to (G_2, \mu_2)$ is a group homomorphism $f : G_1 \to G_2$ (defined over K) such that $\mu_2 = f \circ \mu_1$. Thus, cocharacter data over K define a category $\mathbf{Cochar}_K$. There is a faithful functor

$$\mathbf{Cochar}_{\mathbb{F}_p} \to \mathbf{Zip}$$

defined as follows. Let (G, μ) be a cocharacter datum over $\mathbb{F}_p$. The cocharacter μ gives rise to opposite parabolics P_+, P_- over k, where e.g. $P_+(k)$ consists of elements $g \in G(k)$ such that the limit

$$\lim_{t \to 0} \mu(t) g \mu(t)^{-1}$$

exists. They share a common Levi subgroup $L := P_+ \cap P_- = \mathrm{Cent}_{G_k}(\mu)$. We let $P := P_-, Q := \varphi(P_+), M := \varphi(L)$. Then $\mathcal{Z}_\mu = (G, P, L, Q, M, \varphi)$ is the zip datum associated to (G, μ).

2.1.3 Shimura Data

A major source of cocharacter data (and indeed the main source for this paper) is given by the *Shimura data* (G, X), where G is a reductive group over $\mathbb{Q}$ and X is a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ (where $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ is the Deligne torus) satisfying the following properties:

(i) for all $h \in X$, the Hodge structure on $\mathrm{Lie}(G_{\mathbb{R}})$ defined by $\mathrm{Ad} \circ h$ is of type

$$\{(-1, 1), (0, 0), (1, -1)\}$$

(ii) for all $h \in X$, $\mathrm{ad}(h(i))$ is a Cartan involution of $G^{\mathrm{ad}}_{\mathbb{R}}$

(iii) G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial

A Shimura datum (G, X) gives rise to a cocharacter datum as follows. Fix a homomorphism $h : \mathbb{S} \to G_{\mathbb{R}}$ in X. The complexification of h is a map

$$h_{\mathbb{C}} : \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$$

We define $\mu_h : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$ by $\mu_h(z) = h_{\mathbb{C}}(z, 1)$. In Hodge theoretic terms, h defines a *Hodge structure*, and μ_h defines the corresponding *Hodge filtration* (see Example 2.1).

It is a well-known fact that the $G(\mathbb{C})$-conjugacy class of μ_h contains a cocharacter μ defined over $\overline{\mathbb{Q}}$, and the $G(\overline{\mathbb{Q}})$-conjugacy class of μ is independent of the choice of μ. Furthermore, the $G(\overline{\mathbb{Q}})$-conjugacy class of μ is defined over a number field $E = E(G, X)$, known as the *reflex field*. We thus obtain a cocharacter datum (G, μ) over $\mathbb{Q}$.

Suppose that G is unramified at p, i.e. G is quasi-split over $\mathbb{Q}_p$ and split over an unramified extension of $\mathbb{Q}_p$. Then there exists a hyperspecial subgroup $K_p \subset G(\mathbb{Q}_p)$. Write $\mathcal{G}$ for the flat group scheme over $\mathbb{Z}_p$ such that $\mathcal{G}_{\mathbb{Q}_p} = G$, $\mathcal{G}_{\mathbb{F}_p}$ is a connected reductive group, and $\mathcal{G}(\mathbb{Z}_p) = K_p$ (this is the definition of K_p being hyperspecial). The existence of $\mathcal{G}$ implies that the reflex field E is unramified at p. By conjugating μ if necessary, we may assume μ is defined over $\mathcal{O}_{E_p}$ (since $\mathcal{G}$ is quasi-split) and hence induces a cocharacter of $\mathcal{G}_k$. Thus, when G is unramified at p, we obtain a cocharacter datum $(\mathcal{G}_{\mathbb{F}_p}, \mu)$ over $\mathbb{F}_p$.

Example 2.1. Simple examples of Shimura data are those of *Siegel type*. In this case, $G = \mathrm{GSp}_{2g}$ is the group of symplectic similitudes for some $2g$-dimensional symplectic vector space (V, ψ) over $\mathbb{Q}$ and $X = \mathcal{H}_g^{\pm}$ is the union of the Siegel upper and lower half-spaces.

Suppose ψ is given by the matrix

$$J = \begin{pmatrix} 0 & -I'_g \\ I'_g & 0 \end{pmatrix}$$

where I'_g is the $g \times g$ anti-diagonal matrix with ones on the anti-diagonal, so that for a $\mathbb{Q}$-algebra R,

$$G(R) = \mathrm{GSp}_{2g}(R) = \{g \in \mathrm{GL}_{2g}(R) : g^t J g = cJ \text{ for some } c \in R^{\times}\}$$

Then X may be seen as the $G(\mathbb{R})$-conjugacy class of the homomorphism $h : \mathbb{S} \to G_{\mathbb{R}}$ given by

$$h(a + bi) = a + bJ$$

One can check that the complexification of h is then given by

$$h_{\mathbb{C}}(z_1, z_2) = \begin{pmatrix} z_1 I_g & 0 \\ 0 & z_2 I_g \end{pmatrix}$$

The *Hodge decomposition* of V (induced by h) is given by

$$V_{\mathbb{C}} = \bigoplus_{p,q \in \mathbb{Z} \times \mathbb{Z}} V^{p,q}$$

where $V^{p,q} := \{v \in V_{\mathbb{C}} : h_{\mathbb{C}}(z_1, z_2)v = z_1^{-p} z_2^{-q} v\}$. In this case, the only nonzero summands are $V^{-1,0}$ and $V^{0,-1}$, i.e. the Hodge structure h is of type $\{(-1,0),(0,-1)\}$; in fact, this is true for all conjugates of h.

The cocharacter $\mu : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$, called the *Hodge cocharacter*, is given by

$$\mu(z) = \begin{pmatrix} z I_g & 0 \\ 0 & I_g \end{pmatrix}$$

The *Hodge filtration* of V is given by

$$\operatorname{Fil}^p V = \operatorname{span}\{v \in V_{\mathbb{C}} : \mu(z)v = z^{-r}v \text{ for some } r \geq p\}$$

In this case, the Hodge filtration looks like

$$V_{\mathbb{C}} \supset V^{0,-1} \supset 0$$

Example 2.2. Closely related to the Siegel type Shimura data are the *Hodge type* Shimura data. A Hodge type datum is given by a pair (G, X) such that there exists a symplectic embedding $G \hookrightarrow \mathrm{GSp}_{2g}$ which carries X into $\mathcal{H}_g^{\pm}$. In other words, the Hodge type Shimura data are those that embed into a Siegel type Shimura datum.

Thus, any $h \in X$ gives a Hodge structure of type $\{(-1,0),(0,-1)\}$ for a faithful representation $r : G \to \mathrm{GL}(V)$. Moreover, this is a defining property of Hodge type Shimura data: any Shimura datum (G, X) with a representation $r : G \to \mathrm{GL}(V)$ with central kernel such that $r \circ h$ is of type $\{(-1,0),(0,-1)\}$ for some $h \in X$ is of Hodge type.

Example 2.3. Even more general than the Hodge type Shimura data are the *abelian type* Shimura data. A Shimura datum (G, X) is of abelian type if there exists a Hodge type datum (G_1, X_1) and a central isogeny $G_1^{\mathrm{der}} \to G^{\mathrm{der}}$ which induces an isomorphism of adjoint Shimura data $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) \cong (G^{\mathrm{ad}}, X^{\mathrm{ad}})$.

2.1.4 Stack of G-Zips

Let $\mathcal{Z} = (G, P, L, Q, M, \varphi)$ be a zip datum.

Definition 2.4. A *G-zip of type $\mathcal{Z}$* over a k-scheme S is a tuple $\underline{I} = (I_G, I_P, I_Q, \iota)$ where I_G is a G-torsor over S, I_P is a P-torsor over S, I_Q is a Q-torsor over S, and $\iota : I_P^{(p)}/R_u(P)^{(p)} \to I_Q/R_u(Q)$ is an isomorphism of M-torsors.

Intuitively, the G-torsor I_G describes the extra structure imposed on a k-vector space V, the torsors I_P and I_Q describe partial flags that respect the extra structure, and ι describes certain isomorphisms between the partial flags. The motivating example of this is when V is the de Rham cohomology of an abelian variety $H^1_{\mathrm{dR}}(A/k)$, the partial flags are the Hodge and conjugate filtrations, and ι is given by the Cartier isomorphism (see [Wed08] §1.7 for more details). In fact, this example serves as the basis for the theory of G-zips: the polarization on A makes $H^1_{\mathrm{dR}}(A/k)$ a symplectic vector space, and hence we obtain a GSp_{2g}-zip.

The category of G-zips of type $\mathcal{Z}$ over S is denoted by $G\text{-}\mathbf{Zip}^{\mathcal{Z}}_S$. The categories $G\text{-}\mathbf{Zip}^{\mathcal{Z}}_S$ over k-schemes S give rise to a fibered category $G\text{-}\mathbf{Zip}^{\mathcal{Z}}$ over the category of k-schemes, which is a smooth stack of dimension 0. Furthermore, the stack $G\text{-}\mathbf{Zip}^{\mathcal{Z}}$ can be realized as a quotient stack in the following way. Denote by $x \to \overline{x}$ the projections $P \to L$, $Q \to M$ given by quotienting by the unipotent radical. Define the zip group

$$E_{\mathcal{Z}} := \{(x, y) \in P \times Q : \varphi(\overline{x}) = \overline{y}\}$$

The group $G \times G$ acts on G via $(a, b) \cdot g = agb^{-1}$, and we let $E_{\mathcal{Z}}$ act on G by restricting this action. Letting $U \subset P$ and $V \subset Q$ denote the unipotent radicals, we can decompose $E_{\mathcal{Z}}$ as a semidirect product

$$(U \times V) \rtimes L \xrightarrow{\sim} E_{\mathcal{Z}}, \quad ((u, v), x) \to (xu, \varphi(x)v)$$

The following is Proposition 3.11 in [PWZ11].

Proposition 2.5. *There is an isomorphism of stacks* $G\text{-}\mathbf{Zip}^{\mathcal{Z}} \cong [E_{\mathcal{Z}} \backslash G]$.

The above construction gives rise to a functor ([GK19b] Lemma 1.5.3)

$$\mathbf{Zip} \to \mathbf{Stack}_k$$

which takes a zip datum $\mathcal{Z}$ to the stack $G\text{-}\mathbf{Zip}^{\mathcal{Z}}$. By composing this functor with the functor $\mathbf{Cochar}_{\mathbb{F}_p} \to \mathbf{Zip}$ described above, we may attach to a cocharacter datum (G, μ) a stack of G-zips, which we will denote by $G\text{-}\mathbf{Zip}^{\mu}$.

We say that a morphism of zip data $\mathcal{Z}_1 \to \mathcal{Z}_2$ has central kernel if the underlying group homomorphism $G_1 \to G_2$ has central (scheme-theoretic) kernel. The following important result is Theorem 5.1.1 in [GK19b].

Proposition 2.6. *(Discrete fiber theorem) Let* $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ *be a morphism with central kernel. Then the attached morphism of stacks*

$$\tilde{f} : G_1\text{-}\mathbf{Zip}^{\mathcal{Z}_1} \to G_2\text{-}\mathbf{Zip}^{\mathcal{Z}_2}$$

has discrete fibers on the underlying topological spaces.

2.2 Zip Stratification

2.2.1 Frames of Zip Data

Let $\mathcal{Z} = (G, P, L, Q, M, \varphi)$ be a zip datum. Let (B, T) be a Borel pair in G defined over $\mathbb{F}_p$, by which we mean a Borel subgroup B and maximal torus T, both defined over $\mathbb{F}_p$, with $T \subset B$. A Borel pair exists by Steinberg's Theorem. Write $W := W(G, T)$ for the Weyl group of G_k.

Definition 2.7. Let (B, T) be a Borel pair and $z \in W$. We call (B, T, z) a *W-frame* for $\mathcal{Z}$ if the following conditions are satisfied:

(i) B, T are defined over $\mathbb{F}_p$

(ii) $B \subset P$

(iii) ${}^zB \subset Q$

(iv) $\varphi(B \cap L) = {}^zB \cap M$

If (B, T) is defined over $\mathbb{F}_p$, as it is in our setup, then by the proof of [PWZ11] Proposition 3.7, there exists $z \in W$ such that (B, T, z) is a W-frame.

Let $\Phi \subset X^*(T)$ (resp. $\Phi_L \subset X^*(T)$) be the set of T-roots of G (resp. T-roots of L). We define the positive roots $\Phi^+ \subset \Phi$ by the condition that $\alpha \in \Phi^+$ when the root group $U_{-\alpha} \subset B$. In particular, note that B is generated by T and the *negative* root groups. Write $\Delta \subset \Phi^+$ for the set of positive simple roots. For each $\alpha \in \Phi$, write $s_\alpha \in W$ for the reflection. We thus obtain a Coxeter system $(W, \{s_\alpha, \alpha \in \Delta\})$, and we write $\ell : W \to \mathbb{N}$ for the associated length function.

If P is a parabolic subgroup containing B with Levi subgroup L, we define the *type* $I \subset \Delta$ of P by the condition that $\mathrm{Lie}(L) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_I} \mathfrak{g}_\alpha$, where $\Phi_I = \mathbb{Z}I \cap \Phi$. If P does not contain B, there exists some conjugate parabolic ${}^gP$ containing B, and we define the type of P to be the type of ${}^gP$. Write $\Phi_I^\pm := \Phi_I \cap \Phi^\pm$. If P_-, P_+ are opposite parabolics and P_- is of type I, write $\bar{I}$ for the type of P_+.

Let $I, J \subset \Delta$ be the types of P, Q, respectively. For any subset $K \subset \Delta$, let $W_K \subset W$ be the subgroup generated by $\{s_\alpha, \alpha \in K\}$. Let w_0 (resp. $w_{0,K}$) be the longest element in W (resp. W_K) via the length function ℓ defined above. Define ${}^KW$ (resp. W^K) to be the subset of elements $w \in W$ of minimal length in the coset $W_K w$ (resp. wW_K). Thus, the set ${}^KW$ (resp. W^K) is a set of representatives for the quotient $W_K \backslash W$ (resp. W/W_K). Note that by condition (iv) above, the map $\varphi \circ \mathrm{int}(z)$ (where $\mathrm{int}(z)$ denotes conjugation by z) induces an isomorphism of Coxeter systems

$$\psi : (W_I, I) \xrightarrow{\sim} (W_J, J)$$

Let (G, μ) be a cocharacter datum and let $\mathcal{Z}_\mu = (G, P, L, Q, M, \varphi)$ be the associated zip datum. Fix a Borel pair (B, T) over $\mathbb{F}_p$ such that $B \subset P$. We then have an explicit choice of frame:

Proposition 2.8. *With notation as above, the triple $(B, T, w_0 w_{0,J})$ is a W-frame for $\mathcal{Z}_\mu$.*

Proof. By definition, B, T are defined over $\mathbb{F}_p$ and $B \subset P$. So, we must show that for $z = w_0 w_{0,J}$, ${}^zB \subset Q$ and $\varphi(B \cap L) = {}^zB \cap M$. We will prove both statements by appealing to the Lie algebras.

Recall that P, Q are defined by $P := P_-$, $Q := \varphi(P_+)$ where P_-, P_+ are opposite parabolics of types $I, \bar{I}$, respectively, and we have $J = \varphi(\bar{I})$, $\bar{J} = \varphi(I)$. Write

$$\mathrm{Lie}\, B = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha$$

and

$$\operatorname{Lie} Q = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_{\bar{J}}} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi^+ \setminus \Phi^+_{\bar{J}}} \mathfrak{g}_\alpha$$

The element w_0 takes $\Phi^\pm$ to $\Phi^\mp$ and takes $\Phi^\pm_J$ to $\Phi^\mp_{\bar{J}}$. Meanwhile, $w_{0,J}$ takes $\Phi^\pm_J$ to $\Phi^\mp_J$. Putting all this together, we see that the element $z = w_0 w_{0,J}$ has the following effects:

$$\Phi^-_J \xrightarrow{w_{0,J}} \Phi^+_J \xrightarrow{w_0} \Phi^-_{\bar{J}}$$

$$\Phi^- \setminus \Phi^-_J \xrightarrow{w_{0,J}} \Phi^- \setminus \Phi^-_J \xrightarrow{w_0} \Phi^+ \setminus \Phi^+_{\bar{J}}$$

Thus, we see that

$$\operatorname{Lie}({}^z B) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_{z \cdot \alpha} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^-_{\bar{J}}} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi^+ \setminus \Phi^+_{\bar{J}}} \mathfrak{g}_\alpha$$

Therefore, $\operatorname{Lie}({}^z B) \subset \operatorname{Lie} Q$, hence by the Lie group–Lie algebra correspondence, ${}^z B \subset Q$.

Now we prove the second claim. We see that

$$\operatorname{Lie} L = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_I} \mathfrak{g}_\alpha \quad \text{and} \quad \operatorname{Lie} M = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_{\bar{J}}} \mathfrak{g}_\alpha$$

Thus,

$$\operatorname{Lie}(\varphi(B \cap L)) = \varphi\left(\mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^-_I} \mathfrak{g}_\alpha\right) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^-_{\bar{J}}} \mathfrak{g}_\alpha = \operatorname{Lie}({}^z B \cap M)$$

and therefore $\varphi(B \cap L) = {}^z B \cap M$. $\square$

2.2.2 Stratification

Recall the definition of stratification:

Definition 2.9. A *stratification* of a topological space X is a decomposition $X = \bigsqcup_i X_i$ into locally closed subsets such that the closure of each X_i is a union of X_j.

For $w \in W$, choose a representative $\dot{w} \in N_G(T)$ such that $(w_1 w_2)^\cdot = \dot{w}_1 \dot{w}_2$ whenever $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. This is possible by choosing a Chevalley basis.

Let $\mathcal{Z} = (G, P, L, Q, M, \varphi)$ be a zip datum, and let (B, T, z) be a W-frame of $\mathcal{Z}$. Let $E_{\mathcal{Z}}$ denote the zip group defined in Section 2.1.4. For an element $h \in G$, write $\mathcal{O}_{\mathcal{Z}}(h)$ for the $E_{\mathcal{Z}}$-orbit of h in G, and write $o_{\mathcal{Z}}(h)$ for the corresponding point in $[E_{\mathcal{Z}} \backslash G] \cong G\text{-}\mathbf{Zip}^{\mathcal{Z}}$. For $w \in W$, write $G_w := \mathcal{O}_{\mathcal{Z}}(\dot{w}\dot{z}^{-1})$. By Theorems 7.5 and 11.3 in [PWZ11], the map $w \to G_w$ restricts to two bijections

$$ {}^I W \xrightarrow{\sim} \{E_{\mathcal{Z}}\text{-orbits in } G\} \tag{2.1}$$

$$ W^J \xrightarrow{\sim} \{E_{\mathcal{Z}}\text{-orbits in } G\} \tag{2.2}$$

The G_w are nonsingular subvarieties of G, and provide a disjoint decomposition of G, i.e. $G = \bigsqcup_{w \in {}^I W} G_w$ (see [PWZ11] Theorems 5.10, 5.11). The following dimension formulas are satisfied:

$$\dim(G_w) = \ell(w) + \dim(P) \quad \text{for } w \in {}^I W \cup W^J$$

This provides a stratification of G in the following sense. Let $\psi : (W_I, I) \xrightarrow{\sim} (W_J, J)$ be the isomorphism induced by the W-frame (B, T, z), and write $\leq$ for the Bruhat order on W. We define a relation on ${}^I W$: for $w, w' \in {}^I W$, write $w' \preccurlyeq w$ if there exists a $y \in W_I$ such that $y w' \psi(y)^{-1} \leq w$. This turns out to be a partial order on ${}^I W$. Then we have the following result:

Proposition 2.10. *([PWZ11] Theorem 6.2) For any $w \in {}^I W$, we have*

$$\overline{G}_w = \bigsqcup_{\substack{w' \in {}^I W \\ w' \preccurlyeq w}} G_{w'}$$

We also write $\mathcal{X}_w := [E_{\mathcal{Z}} \backslash G_w]$ (resp. $\overline{\mathcal{X}}_w := [E_{\mathcal{Z}} \backslash \overline{G}_w]$) for the open (resp. closed) strata of the stack $[E_{\mathcal{Z}} \backslash G] \cong G\text{-}\mathbf{Zip}^{\mathcal{Z}}$.

2.3 Hasse Invariants on the Stack of G-Zips

2.3.1 Characteristic Sections on Strata

Recall that the classical Hasse invariant is a mod p modular form, viewed as a section of a certain line bundle on the special fiber of the modular curve, and it is nonzero precisely on the ordinary locus. We generalize this property with the following definition.

Definition 2.11. Let S be locally closed in a scheme (or algebraic stack) X, with Zariski closure $\overline{S}$ (endowed with reduced structure), and let $\mathscr{L}$ be a line bundle on X. We say that a section $f \in H^0(\overline{S}, \mathscr{L})$ is a *characteristic section* for S if its non-vanishing locus in $\overline{S}$ is exactly S.

The existence of characteristic sections on the stack of G-zips is of primary interest, since these will be our group-theoretic Hasse invariants. Thus, we want to know when a line bundle admits a characteristic section, or if such a line bundle even exists.

Definition 2.12. We call a stratification of a scheme (or stack) X *principally pure* if, for each stratum S, there exists a line bundle $\mathscr{L}$ on X and a characteristic section $f \in H^0(\overline{S}, \mathscr{L})$ for S. Furthermore, we call the stratification *uniformly principally pure* if $\mathscr{L}$ may be chosen independently of S.

Fix a zip datum $\mathcal{Z} = (G, P, L, Q, M, \varphi)$, and assume there exists a torus $T \subset L$ defined over $\mathbb{F}_p$ (this can be arranged by conjugating the zip datum). Denote the zip group $E := E_{\mathcal{Z}}$ (see section 2.1.4). Identify $X^*(E) = X^*(L)$ via the projections $E \to P \to L$. For a character $\chi \in X^*(L)$, denote by $\mathscr{L}_{\mathrm{Zip}}(\chi)$ the line bundle on $[E \backslash G] \cong G\text{-}\mathbf{Zip}^{\mathcal{Z}}$ such that

$$H^0([E \backslash G], \mathscr{L}_{\mathrm{Zip}}(\chi)) = \{ f : G \to \mathbb{A}^1 \mid f(e \cdot g) = \chi(e) f(g) \quad \forall e \in E, g \in G \}$$

Similarly, denote by $\mathcal{L}(\chi)$ the line bundle on G/P such that

$$H^0(G/P, \mathcal{L}(\chi)) = \{ f : G \to \mathbb{A}^1 \mid f(gp) = \chi(p)^{-1} f(g) \quad \forall p \in P, g \in G \}$$

Definition 2.13. Let $\chi \in X^*(T)$. We say that χ is *orbitally p-close* if $\left|\frac{\langle \chi, \sigma\alpha^\vee \rangle}{\langle \chi, \alpha^\vee \rangle}\right| \leq p-1$ for every $\alpha \in \Phi$ satisfying $\langle \chi, \alpha^\vee \rangle \neq 0$, and for all $\sigma \in W \rtimes \mathrm{Gal}(k/\mathbb{F}_p)$.

Definition 2.14. A character $\chi \in X^*(L)$ is *L-ample* if the associated line bundle $\mathcal{L}(\chi)$ on G/P is anti-ample.

Remark 2.15. Let (B, T, z) be a W-frame for $\mathcal{Z}$. The following are equivalent:

(i) $\chi \in X^*(L)$ is L-ample

(ii) $\langle \chi, \alpha^\vee \rangle < 0$ for all $\alpha \in \Delta \setminus I$, where I is the type of P

For a proof, see [Jan03] Proposition II.4.4 and Remark 1 immediately following the proposition.

By [KW18] Proposition 3.3.1, the space $H^0([E\backslash G_w], \mathscr{L}_{\mathrm{Zip}}(\chi))$ is at most 1-dimensional for all $\chi \in X^*(L)$ and $w \in {}^IW$. By [Kos16] Theorem 3.1, we can fix an integer $N \geq 1$ such that for all $\chi \in X^*(L)$ and $w \in {}^IW$, the space $H^0([E\backslash G_w], \mathscr{L}_{\mathrm{Zip}}(N\chi))$ has dimension exactly 1 over k. For each $w \in {}^IW$, denote by $h_{w,\chi}$ (or simply h_w) a nonzero element of $H^0([E\backslash G_w], \mathscr{L}_{\mathrm{Zip}}(N\chi))$. Then we have the following result:

Proposition 2.16. *([GK19a] Theorem 3.2.3) If $\chi \in X^*(L)$ is L-ample and orbitally p-close, then there exists $d \geq 1$ such that for all $w \in {}^IW$, the section $h_{w,\chi}^d$ extends to $[E\backslash \overline{G}_w]$ with nonvanishing locus $[E\backslash G_w]$, i.e. $h_{w,\chi}^d$ is a characteristic section for $[E\backslash G_w]$.*

Thus, to show that the stack G-$\mathbf{Zip}^{\mathcal{Z}}$ is uniformly principally pure, it suffices to show that L admits a character $\chi \in X^*(L)$ that is L-ample and orbitally p-close, in which case we call χ a *Hasse generator* of G-$\mathbf{Zip}^{\mathcal{Z}}$. This is not always the case (see [GK19b] §4.3), but when it is, we call the resulting characteristic sections *group theoretic Hasse invariants.*

The next definition will be used in Theorem 4.4 (iii).

Definition 2.17. Let $\delta \in \mathbb{R}_{\geq 0}$. We say that $\chi \in X^*(T)$ is *δ-regular* if $|\langle \chi, \alpha^\vee \rangle| > \delta$ for all $\alpha \in \Phi$.

2.3.2 The Griffiths Character

Following [Gol19], we introduce the Griffiths character of a cocharacter datum (G, μ) and show that it is a Hasse generator of G-$\mathbf{Zip}^{\mu}$.

Let (G, μ) be a cocharacter datum over a field K as in §2.1.2. Let $r : G \to \mathrm{GL}(V)$ be a morphism with central kernel. The cocharacter $r \circ \mu$ of $\mathrm{GL}(V)$ induces a descending filtration $\mathrm{Fil}^\bullet V$ on V given by

$$\mathrm{Fil}^a V = \bigoplus_{a' \geq a} V_{-a'}$$

where $V_b = \{v \in V : (r \circ \mu)(z)v = z^b v\}$ is the b-weight space of $r \circ \mu$ acting on V. Let $(r \circ \mu)_{\max}$ (resp. $(r \circ \mu)_{\min}$) be the largest (resp. smallest) $(r \circ \mu)$-weight in V, so that

$$\mathrm{Fil}^{-(r\circ\mu)_{\max}} V = V, \qquad \mathrm{Fil}^{1-(r\circ\mu)_{\min}} V = 0$$

Define the *Griffiths module* of (G,μ,r) to be

$$\mathrm{Grif}(G,\mu,r) := \sum_{a=1-(r\circ\mu)_{\max}}^{-(r\circ\mu)_{\min}} \mathrm{Fil}^a V$$

and the *Griffiths character* to be

$$\mathrm{grif}(G,\mu,r) := \det \mathrm{Grif}(G,\mu,r)$$

With $L = \mathrm{Cent}_{G_{\overline{K}}}(\mu)$, we see that $\mathrm{Grif}(G,\mu,r)$ is an L-module and $\mathrm{grif}(G,\mu,r) \in X^*(L)$.

In order to state the main result of [Gol19] about the Griffiths character, we need a bit of group-theoretic setup. Denote by G^{der} (resp. G^{ad}, $\tilde{G}$) the derived subgroup of G (resp. its adjoint quotient, the simply-connected covering of G^{der}). Let $s : \tilde{G} \to G$ be the natural quasi-section of the projection $\mathrm{pr} : G \to G^{\mathrm{ad}}$, i.e. s is $\tilde{G} \twoheadrightarrow G^{\mathrm{der}} \hookrightarrow G$.

The root datum of (G,T,Δ) as in §2.2.1 induce root data for $G^{\mathrm{der}}, G^{\mathrm{ad}}, \tilde{G}$ as follows. Let

$$X_0^*(T) := \{\chi \in X^*(T) : \langle \chi, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in \Phi\}$$

Define a compatible maximal torus $T^{\mathrm{der}} \subset G^{\mathrm{der}}$ by

$$T^{\mathrm{der}} := \bigcap_{\chi \in X_0^*(T)} \ker \chi$$

It has character group $X^*(T^{\mathrm{der}}) = X^*(T)/X_0^*(T)$, and the roots (resp. simple roots) of $(G^{\mathrm{der}}, T^{\mathrm{der}})$ are given by restriction of Φ (resp. Δ). Now, define T^{ad} (resp. $\tilde{T}$) as the image of T^{der} in G^{ad} (resp. the preimage of T^{der} in $\tilde{G}$). The root data of $(G^{\mathrm{der}}, T^{\mathrm{der}})$, $(G^{\mathrm{ad}}, T^{\mathrm{ad}})$, $(\tilde{G}, \tilde{T})$ are identified via the central isogenies

$$(\tilde{G}, \tilde{T}) \to (G^{\mathrm{der}}, T^{\mathrm{der}}) \to (G^{\mathrm{ad}}, T^{\mathrm{ad}})$$

These isogenies give canonical identifications

$$X^*(\tilde{T})_{\mathbb{Q}} \cong X^*(T^{\mathrm{der}})_{\mathbb{Q}} \cong X^*(T^{\mathrm{ad}})_{\mathbb{Q}} \text{ and } X_*(\tilde{T})_{\mathbb{Q}} \cong X_*(T^{\mathrm{der}})_{\mathbb{Q}} \cong X_*(T^{\mathrm{ad}})_{\mathbb{Q}}$$

and hence their Weyl groups are all isomorphic to W, the Weyl group of G. Choose a positive definite, symmetric, $W \rtimes \mathrm{Gal}(\overline{K}/K)$-invariant bilinear form

$$(\,,\,) : X^*(\tilde{T})_{\mathbb{Q}} \times X_*(\tilde{T})_{\mathbb{Q}} \to \mathbb{Q}$$

When G^{ad} is K-simple, such a form is unique up to positive scalar. The triple $(X^*(\tilde{T})_{\mathbb{Q}}, \Phi, (\,,\,))$ is then a root system associated to (G,T). Write

$$\overline{\mathrm{grif}}(G,\mu,r) := \mathbb{Q}_{>0} s^* \mathrm{grif}(G,\mu,r)$$

for the positive ray generated by $s^* \mathrm{grif}(G,\mu,r)$ in $X^*(\tilde{T})_{\mathbb{Q}}$. This is called the *Griffiths ray.*

Given a root $\gamma \in \Phi$ and a representation $r : G \to \mathrm{GL}(V)$, let

$$S(\gamma^\vee, r) := \sum_{\chi \in \Phi(V,T)} m_V(\chi) \langle \chi, \gamma^\vee \rangle^2$$

where $\Phi(V,T)$ is the set of $T_{\overline{K}}$-weights of $V_{\overline{K}}$ and $m_V(\chi)$ is the dimension of the χ-weight space of V. Note that $S(\gamma^\vee, r)$ is invariant under simple reflections.

The main result of [Gol19] is the following:

Proposition 2.18. *([Gol19] Theorem 3.1) Let (G,μ) be a cocharacter datum over a field K, and assume G^{ad} is K-simple. Let $r: G \to \mathrm{GL}(V)$ be a representation of G over K with central kernel. Then*

(i) For all $\alpha \in \Delta$, one has

$$\langle \mathrm{grif}(G,\mu,r), \alpha^\vee \rangle = -\frac{1}{2}\langle \alpha, \mu \rangle S(\alpha^\vee, r)$$

(ii) The value $(\alpha,\alpha)S(\alpha^\vee, r)$ is independent of $\alpha \in \Phi$

(iii) Under the identification $X^*(\tilde{T})_{\mathbb{Q}} \cong X_*(\tilde{T})_{\mathbb{Q}}$ afforded by $(,)$, for every $\alpha \in \Phi$ one has

$$s^* \,\mathrm{grif}(G,\mu,r) = -\frac{(\alpha,\alpha)}{4} S(\alpha^\vee, r)\mu^{ad}$$

where $\mu^{ad} = \mathrm{pr} \circ \mu$ is the projection of μ onto $G^{ad}_{\overline{K}}$.

(iv) In particular,

$$\overline{\mathrm{grif}}(G,\mu,r) = -\langle \mu^{ad} \rangle \qquad \text{in } X^*(\tilde{T})_{\mathbb{Q}}$$

Thus, the Griffiths ray $\overline{\mathrm{grif}}(G,\mu,r)$ is independent of r, and up to central isogeny, the Griffiths character is given by $-\mu$. When the cocharacter datum (G,μ) comes from a Hodge type Shimura datum, we recover the *Hodge character*, i.e. the character giving rise to the Hodge bundle, as the following example shows.

Example 2.19. Consider a Hodge type cocharacter datum (G,μ) and let $r: G \hookrightarrow \mathrm{GSp}(V,\psi)$ be the symplectic embedding (see Example 2.2). As we saw in Example 2.1, the filtration of V induced by $r \circ \mu$ is the Hodge filtration

$$V_{\mathbb{C}} \supset V^{0,-1} \supset 0$$

Thus, the Griffiths module of (G,μ,r) is given by

$$\mathrm{Grif}(G,\mu,r) = V^{0,-1}$$

and hence the Griffiths character is the representation

$$\mathrm{grif}(G,\mu,r) = \det V^{0,-1}$$

of $L := \mathrm{Cent}_{G_{\mathbb{C}}}(\mu)$. This character gives rise to the Hodge bundle (see Example 3.6).

Corollary 2.20. *Let (G,μ) be a cocharacter datum with the assumptions from Proposition 2.18. If μ is orbitally p-close, then* $\mathrm{grif}(G,\mu,r)$ *is a Hasse generator.*

Proof. Since μ is orbitally p-close, so is $-\mu^{\mathrm{ad}}$, and hence by Proposition 2.18 (iii), $\mathrm{grif}(G,\mu,r)$ is also orbitally p-close.

Thus, by Proposition 2.16 it remains to show that $\mathrm{grif}(G,\mu,r)$ is L-ample. Let (B,T,z) be a W-frame and let $I \subset \Delta$ be the type of the parabolic P as in Section 2.2.1. Since μ is dominant with respect to Δ, we see that $\langle \alpha, \mu \rangle \geq 0$ for all $\alpha \in \Delta$ with equality if and only if $\alpha \in I$. Thus, by Proposition 2.18 (i), we have that

$$\langle \mathrm{grif}(G,\mu,r), \alpha^\vee \rangle < 0 \text{ for all } \alpha \in \Delta \setminus I$$

By Remark 2.15, this proves that $\mathrm{grif}(G,\mu,r)$ is L-ample. ∎

2.3.3 The Griffiths Bundle on a Zip Scheme

Let (G, μ) be a cocharacter datum and $r : G \to \mathrm{GL}(V)$ a homomorphism with central kernel. Denote by $\mathscr{G} := \mathscr{L}_{\mathrm{Zip}}(\mathrm{grif}(G, \mu, r))$ the line bundle on G-$\mathbf{Zip}^\mu$ corresponding the Griffiths character. By the results of the previous section, $\mathscr{G}$ admits group-theoretic Hasse invariants on zip strata.

We will be interested in pulling back the Griffiths line budle to a zip scheme. For $\mathcal{Z}$ a zip datum with underlying group G, we call a scheme X a *zip scheme* if there exists a smooth surjective morphism $\zeta : X \to G\text{-}\mathbf{Zip}^{\mathcal{Z}}$. A major question in the study of zip schemes is: to what extent is the geometry of X controlled by G-$\mathbf{Zip}^{\mathcal{Z}}$ and ζ? More specifically for our purposes, what can we say about a vector bundle on X given by a pullback via ζ? Some answers in this direction are addressed in the upcoming paper [BGKS], such as the following:

Proposition 2.21. *Let $\zeta : X \to G\text{-}\mathbf{Zip}^\mu$ be a projective zip scheme, and let $\mathscr{F}$ be a line bundle on G-$\mathbf{Zip}^\mu$ corresponding to a Hasse generator. Then the line bundle $\mathscr{F}_X := \zeta^*\mathscr{F}$ is nef and, moreover, ample.*

Proof. Nefness is proven in the proof of Corollary 3.7 in [Gol19], but we repeat the proof here for clarity. Ampleness is a main result of [BGKS], so we direct the reader to that paper.

Let $Z \subset X$ be a closed, integral subscheme. To show nefness, we will construct a nonzero section h_Z in $H^0(Z, \mathscr{F}_X)$. Since Z is irreducible, there exists a unique stratum $\mathcal{X}_w$ in G-$\mathbf{Zip}^\mu$ such that $\zeta^*\mathcal{X}_w \cap Z$ is dense in Z. Then we may take $h_Z := \zeta^* h_w$. □

In particular, the pullback $\mathscr{G}_X := \zeta^*\mathscr{G}$ is ample on a projective zip scheme X.

2.3.4 Zip Flags and the Cone of Global Sections

Continue to let (G, μ) be a cocharacter datum, and let (B, T) be a Borel pair as in §2.2.1. We now introduce the stack of G-zip flags, which will be used in the proof of Theorem 4.4 (ii).

Definition 2.22. A *G-zip flag of type μ* over a k-scheme S is a a pair $\hat{I} = (\underline{I}, J)$ where $\underline{I} = (I_G, I_P, I_Q, \iota)$ is a G-zip of type μ over S and $J \subset I_P$ is a B-torsor.

Recall the intuitive explanation of G-zip in §2.1.4. We can similarly reason about G-zip flags: the additional data of J describes a complete flag which extends the partial flag given by I_P. In terms of Hodge theory, we can think about this as "filling out" the Hodge filtration.

We denote by G-$\mathbf{ZipFlag}^\mu(S)$ the category of G-zip flags over S, and similarly to G-zips, we obtain a stack G-$\mathbf{ZipFlag}^\mu$ over k, the stack of G-zip flags of type μ. This is independent of (B, T) up to isomorphism. There is a natural projection

$$\pi : G\text{-}\mathbf{ZipFlag}^\mu \to G\text{-}\mathbf{Zip}^\mu$$

which by [GK19a] Theorem 2.1.2 satisfies the following commutative diagram

$$\begin{tikzcd} G\text{-}\mathbf{ZipFlag}^\mu \arrow[r, "\pi"] \arrow[d, "\cong"] & G\text{-}\mathbf{Zip}^\mu \arrow[d, "\cong"] \\ {[(E \times B)\backslash(G \times P)]} \arrow[r, "\pi"] & {[E\backslash G]} \end{tikzcd}$$

where the action of $E \times B$ on $G \times P$ is given by $((a,b),c)\cdot(g,r) = (agb^{-1}, arc^{-1})$ where $(a,b) \in E$, $c \in B$, and $(g,r) \in G \times P$. By defining $E' := E \cap (B \times G)$, one can see that $E' \subset B \times {}^{z}B$ and the map $G \to G \times P$ given by $g \to (g,1)$ induces an isomorphism of stacks

$$[E' \backslash G] \cong [(E \times B)\backslash(G \times P)] \cong G\text{-}\mathbf{ZipFlag}^{\mu}$$

We will also need the notion of the Schubert stack, which we now briefly introduce (see [GK19a] §2.2 and §2.3 for more details). Let $B \times B$ act on G by $(a,b)\cdot g = agb^{-1}$ for $a,b \in B$ and $g \in G$. Define the Schubert stack by $\mathrm{Sbt} := [(B \times B)\backslash G]$. By [GK19a] §2.3, there exists a smooth morphism of stacks

$$\psi : G\text{-}\mathbf{ZipFlag}^{\mu} \to \mathrm{Sbt}$$

We obtain the following diagram:

$$\begin{tikzcd} G\text{-}\mathbf{ZipFlag}^{\mu} \arrow[r, "\psi"] \arrow[d, "\pi"] & \mathrm{Sbt} \\ G\text{-}\mathbf{Zip}^{\mu} & \end{tikzcd}$$

Recall from §2.3.1 that a character $\lambda \in X^*(L)$ gives rise to a line bundle $\mathscr{L}_{\mathrm{Zip}}(\lambda)$ on G-$\mathbf{Zip}^{\mu}$. Similarly, by identifying $X^*(E') = X^*(B) = X^*(T)$ via the first projection $E' \to B$ and inclusion $T \subset B$, one can see that a character $\lambda \in X^*(T)$ gives rise to a line bundle $\mathscr{L}_{\mathrm{Flag}}(\lambda)$ on G-$\mathbf{ZipFlag}^{\mu}$. In the same way, for each pair $(\lambda, \mu) \in X^*(T) \times X^*(T)$ we have a line bundle $\mathscr{L}_{\mathrm{Sbt}}(\lambda,\mu)$ on Sbt. By [GK19a] Lemma 3.1.1, these are related in the above diagram via

$$\pi^*\mathscr{L}_{\mathrm{Zip}}(\lambda) = \mathscr{L}_{\mathrm{Flag}}(\lambda) \quad \text{and} \quad \psi^*\mathscr{L}_{\mathrm{Sbt}}(\lambda,\mu) = \mathscr{L}_{\mathrm{Flag}}(\lambda + p\,{}^{\sigma}(z\mu))$$

where $\sigma : k \to k$ is the inverse of $x \to x^p$.

Recall that w_0 denotes the longest element in W. By [GK19a] Lemma 3.4.1, we obtain an important subclass of character pairs in $X^*(T) \times X^*(T)$ which give rise to global sections on the Schubert stack:

$$H^0(\mathrm{Sbt}, \mathscr{L}_{\mathrm{Sbt}}(\lambda, -w_0\lambda)) \neq 0 \iff -\lambda \in X^*_+(T)$$

where $X^*_+(T)$ denotes the dominant characters (with respect to B). By the above pullback relations, we have

$$\psi^*\mathscr{L}_{\mathrm{Sbt}}(\lambda, -w_0\lambda) = \mathscr{L}_{\mathrm{Flag}}(\lambda - p\,{}^{\sigma}(zw_0\mu))$$

hence we define a map

$$D_{w_0} : X^*(T) \to X^*(T), \quad \lambda \to \lambda - p\,{}^{\sigma}(zw_0\mu)$$

Define the subset $\mathcal{C}_{w_0} \subset X^*(T)$ by

$$\mathcal{C}_{w_0} := \left\{ D_{w_0}(\lambda) : -\lambda \in X^*_+(T) \right\}$$

which is a cone of maximal rank by [GK19a] Lemma 3.4.2. By the above characterization of global sections on Sbt, we can see that for any $\chi \in \mathcal{C}_{w_0}$, $\mathscr{L}_{\mathrm{Flag}}(\chi)$ admits a nonzero global section (via pullback by ψ).

Denote by $\mathscr{G}$ the Griffiths bundle on G-$\mathbf{Zip}^\mu$. Recall that by [Kos16] Theorem 3.1 we can fix an integer $N \geq 1$ so that $\dim H^0([E\backslash G_{w_0}], \mathscr{G}^N) = 1$. Define another cone $\mathcal{C}$ by

$$\mathcal{C} := \mathbb{N}(N \operatorname{grif}(G, \mu, r)) + \mathcal{C}_{w_0}$$

That is, $\mathcal{C}$ is the cone generated by $\mathcal{C}_{w_0}$ and $N \operatorname{grif}(G, \mu, r)$. The main result of this section is the following, which was proven in [GK19a] Proposition 3.4.3 for Hodge type cocharacter data. Here, we modify the proof slightly so that it works for abelian type cocharacter data.

Proposition 2.23. *Let (G, μ) be an abelian type cocharacter datum. For any $\chi \in \mathcal{C}$, there exists a nonzero global section $h_\chi \in H^0(G\text{-}\mathbf{ZipFlag}^\mu, \mathscr{L}_{Flag}(\chi))$.*

Proof. We already saw above that any $\chi \in \mathcal{C}_{w_0}$ admits a global section, so it suffices to prove this for $N \operatorname{grif}(G, \mu, r)$, i.e. that $\mathscr{G}^N$ admits a nonzero global section. Since $\operatorname{grif}(G, \mu, r)$ is a Hasse generator by Corollary 2.20, there exists a group theoretic Hasse invariant for the unique open stratum $[E\backslash G_{w_0}]$ of G-$\mathbf{Zip}^\mu$, i.e. for some $d \geq 1$ there exists a section $h \in H^0([E\backslash \overline{G}_{w_0}], \mathscr{G}^d) = H^0(G\text{-}\mathbf{Zip}^\mu, \mathscr{G}^d)$ with nonvanishing locus $[E\backslash G_{w_0}]$. Thus, h is a nonzero global section of $\mathscr{G}^d$ over G-$\mathbf{Zip}^\mu$.

To show that $\mathscr{G}^N$ admits a nonzero global section too, let $f \in H^0([E\backslash G_{w_0}], \mathscr{G}^N) \neq 0$ be nonzero. Then f^d and h^N are both sections of $\mathscr{G}^{Nd}$ over the open stratum $[E\backslash G_{w_0}]$, and thus they must coincide up to a scalar since $\dim H^0([E\backslash G_{w_0}], \mathscr{G}^N) = 1$. Therefore, f^d extends to G (when viewed as a rational function of G), and since G is a normal scheme, f must also extend to G. Thus, $\mathscr{G}^N$ has a nonzero global section over G-$\mathbf{Zip}^\mu$. □

3 Hasse Invariants for Proper Abelian Type Shimura Varieties

In this section, we apply the result of the previous section to abelian type Shimura varieties.

3.1 Shimura Varieties

3.1.1 Definition and Examples

Let (G, X) be a Shimura datum with reflex field $E := E(G, X)$ (see §2.1.3). For a (sufficiently small) compact open subgroup $K \subset G(\mathbb{A}^\infty)$, the *Shimura variety* $\mathrm{Sh}_K(G, X)$ corresponding to the Shimura datum (G, X) is a variety over E whose complex points are given by

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}^\infty)/K$$

The inverse system of Shimura varieties $(\mathrm{Sh}_K(G, X))_K$, where K runs over all (sufficiently small) compact open subgroups of $G(\mathbb{A}^\infty)$, is endowed with an action by $G(\mathbb{A}^\infty)$ as follows. Let $g \in G(\mathbb{A}^\infty)$. Then $K \to g^{-1}Kg$ maps open compact subgroups to open compact subgroups, and we obtain a map

$$g : \mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{g^{-1}Kg}(G, X)$$

given on complex points by

$$[x, a] \to [x, ag]$$

Every inclusion $K' \subset K$ induces a projection

$$\pi_{K'/K} : \mathrm{Sh}_{K'}(G, X) \to \mathrm{Sh}_K(G, X)$$

Let $G(\mathbb{R})_+$ denote the subgroup of $G(\mathbb{R})$ whose image in $G^{\mathrm{ad}}(\mathbb{R})$ lies in the connected component $G^{\mathrm{ad}}(\mathbb{R})^+$, and let $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$. Let $\mathcal{C}(G)$ denote a set of representatives for the finite set $G(\mathbb{Q})_+ \backslash G(\mathbb{A}^\infty)/K$ and let X^+ be a connected component of X. Then

$$\mathrm{Sh}_K(G, X)_{\mathbb{C}} = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}^\infty)/K \cong \bigsqcup_{g \in \mathcal{C}(G)} \Gamma_g \backslash X^+$$

where Γ_g is the image of $gKg^{-1} \cap G(\mathbb{Q})_+ \subset G(\mathbb{Q})_+$ in $G^{\mathrm{ad}}(\mathbb{R})^+$. This shows how Shimura varieties can be decomposed into a union of *connected Shimura varieties*, which are often much simpler to work with.

Example 3.1. When (G, X) is a Siegel type datum (see Example 2.1), the Shimura variety $\mathrm{Sh}_K(G, X)$ is called a *Siegel modular variety.* In particular, when $G = \mathrm{GL}_2$ and $X = \mathcal{H}_1^{\pm}$ is the union of the upper and lower complex half planes, letting $K(N) = \ker(\mathrm{GL}_2(\hat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}))$ gives the *modular curve*

$$\mathrm{Sh}_{K(N)}(\mathrm{GL}_2, \mathcal{H}_1^{\pm}) = \bigsqcup_{(\mathbb{Z}/N\mathbb{Z})^\times} Y(N)$$

where $Y(N) = \Gamma(N) \backslash \mathcal{H}_1$ is the classical modular curve of level N.

Recall that $Y(N)$ is a complex manifold which has a canonical model over the number field $\mathbb{Q}(\zeta_N)$; in fact, $Y(N)$ is an example of a connected Shimura variety. This model relies on a choice of identification $\mu_N \cong \mathbb{Z}/N\mathbb{Z}$. Thus, by taking a disjoint union over all such identifications, i.e. over $(\mathbb{Z}/N\mathbb{Z})^\times$, we obtain a model over $\mathbb{Q}$; that is, the reflex field of $(\mathrm{GL}_2, \mathcal{H}_1^{\pm})$ is $\mathbb{Q}$. This is one main benefit of considering non-connected Shimura varieties.

Example 3.2. More generally, for a Hodge type datum (G, X) (see Example 2.2), the resulting Shimura variety $\mathrm{Sh}_K(G, X)$ is called a *Hodge type Shimura variety.* These Shimura varieties are known to have a modular interpretation: given the symplectic embedding $G \hookrightarrow \mathrm{GSp}(V, \psi)$, there exist multilinear maps $t_i : V \times \cdots \times V \to \mathbb{Q}$ for $i = 1, \ldots, n$ such that G is precisely the subgroup of $\mathrm{GSp}(V, \psi)$ fixing the t_i. Then $\mathrm{Sh}_K(G, X)$ parametrizes isomorphism classes of triples $(A, (s_i)_{0 \le i \le n}, \eta K)$ where

- A is an abelian variety
- s_0 (or $-s_0$) is a polarization for $H_1(A, \mathbb{Q})$
- $s_1, \ldots, s_n$ are certain (Hodge) tensors on $H_1(A, \mathbb{Q})$
- ηK is a K-orbit of $\mathbb{A}^\infty$-linear isomorphisms $V(\mathbb{A}^\infty) \to V_f(A)$ (the rational Tate module of A) sending ψ to s_0 and t_i to s_i

Example 3.3. Even more generally, for an abelian type datum (G, X) (see Example 2.3), we obtain the *abelian type Shimura varieties* $\mathrm{Sh}_K(G, X)$. While these Shimura varieties are not known in general to have a modular interpretation, one often works with them via their relation to the Hodge type Shimura varieties. Specifically, let (G_1, X_1) be an associated Hodge type Shimura datum, i.e. there exists a central isogeny $\varphi : G_1^{\mathrm{der}} \to G^{\mathrm{der}}$ which induces an isomorphism of adjoint Shimura data $\varphi : (G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) \cong (G^{\mathrm{ad}}, X^{\mathrm{ad}})$. We may compatibly choose level subgroups $K \subset G(\mathbb{A}^\infty), K_1 \subset G_1(\mathbb{A}^\infty)$ so that $K^{\mathrm{ad}} = K_1^{\mathrm{ad}}$, where K^{ad} is the image of K under the adjoint map (similarly for K_1). This gives a commutative diagram

$$\begin{tikzcd}
\mathrm{Sh}_{K_1}(G_1, X_1) \arrow[dr, "\pi_1^{\mathrm{ad}}"] & \\
& \mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \\
\mathrm{Sh}_K(G, X) \arrow[ur, "\pi^{\mathrm{ad}}"'] &
\end{tikzcd}$$

This is merely one example of the relation between abelian type and Hodge type Shimura varieties. For further examples, see [SZ22] (where the above adjoint relation is used) and [Lov17b] (who introduces some auxiliary Shimura data, i.e. see the diagram in §4.7.2).

Note that Shimura varieties are not proper in general, but there is a simple condition for properness. Recall that an algebraic group G is *Q-anisotropic* if it does not contain any nontrivial $\mathbb{Q}$-split tori. The key result is the following:

Theorem 3.4. (Borel's Theorem) *Let (G, X) be a Shimura datum. Then $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ is compact if and only if G^{der} is $\mathbb{Q}$-anisotropic.*

3.1.2 Integral Canonical Models

Let (G, X) be a Shimura datum and assume that G is unramified at p, i.e. quasi-split at p and split over an unramified extension of $\mathbb{Q}_p$. Recall that the Shimura variety $\mathrm{Sh}_K(G, X)$ has a model over its reflex field E. Let $\mathcal{O}_E$ denote its ring of integers, and let $\mathfrak{p} \mid p$ be a prime of $\mathcal{O}_E$. Let $K_p \subset G(\mathbb{A}^\infty)$ be a *hyperspecial subgroup*, i.e. a subgroup such that $K_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ where $G_{\mathbb{Z}_p}$ is a reductive group over $\mathbb{Z}_p$ with generic fiber G (we will abuse notation and simply write G for $G_{\mathbb{Z}_p}$). Hyperspecial subgroups exist if and only if G is unramified at p. Langlands suggested that the tower of E-schemes

$$\mathrm{Sh}_{K_p}(G, X) = \varprojlim_{K_p} \mathrm{Sh}_{K_p K^p}(G, X)$$

over subgroups $K^p \subset G(\mathbb{A}^{\infty,p})$ should admit a $G(\mathbb{A}^{\infty,p})$-equivariant extension to a tower of $\mathcal{O}_{E,\mathfrak{p}}$-schemes $\mathscr{S}_{K_p}(G, X)$ (where $\mathcal{O}_{E,\mathfrak{p}}$ is the localization at $\mathfrak{p}$).

Of course, without any further conditions on the integral model, one could simply take $\mathscr{S}_{K_p}(G, X) = \mathrm{Sh}_{K_p}(G, X)$. Milne suggested that $\mathscr{S}_{K_p}(G, X)$ should satisfy the *extension property*: a scheme or pro-scheme X over $\mathcal{O}_{E,\mathfrak{p}}$ satisfies the extension property if for any regular, formally smooth $\mathcal{O}_{E,\mathfrak{p}}$-scheme S, a map $S \otimes E \to X$ extends to S. A smooth $G(\mathbb{A}^{\infty,p})$-equivariant model $\mathscr{S}_{K_p}(G, X)$ of $\mathrm{Sh}_{K_p}(G, X)$ over $\mathcal{O}_{E,\mathfrak{p}}$ which satisfies the extension property is called an *integral canonical model*. Given an integral canonical model $\mathscr{S}_{K_p}(G, X)$, one can obtain an integral canonical model of $\mathrm{Sh}_K(G,X)$ for $K = K_pK^p$ by taking the quotient: $\mathscr{S}_K(G,X) := \mathscr{S}_{K_p}(G,X)/K^p$.

Integral canonical models were shown to exist for abelian type Shimura data by Kisin in [Kis10]. In [Lov17b], Lovering shows that these models can be spread out to smooth models over $\mathcal{O}_E[1/N]$ where $N > 1$ is an integer divisible by the ramified primes of G. More generally, the recent work of Bakker, Shankar, and Tsimerman ([BST24]) shows that all Shimura varieties admit integral canonical models for sufficiently large primes p. Notably, this includes exceptional Shimura varieties, and recovers the work of Kisin in the abelian type case.

3.1.3 Automorphic Vector Bundles

An important class of vector bundles on a Shimura variety $\mathrm{Sh}_K(G,X)$ are the *automorphic vector bundles*. We follow the definition of automorphic bundle given by Milne in [Mil90]. We fix a point $h \in X$. Recall from §2.1.3 we can associate to h a cocharacter $\mu : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$, and μ defines a decreasing filtration $\mathrm{Fil}(\mu)$ on the category $\mathbf{Rep}_{\mathbb{C}}(G)$ of representations of G. Define the *compact dual* $\check{X}$ of X to be the $G(\mathbb{C})$-conjugacy class of filtrations of $\mathbf{Rep}_{\mathbb{C}}(G)$ containing $\mathrm{Fil}(\mu)$. Let $P \subset G_{\mathbb{C}}$ denote the parabolic subgroup which fixes $\mathrm{Fil}(\mu)$. Then

$$\check{X} \cong G(\mathbb{C})/P(\mathbb{C})$$

Moreover, $\check{X}$ can be realized as a subvariety of a Grassman variety, and is in fact defined over the reflex field E.

The map

$$\beta : X \to \check{X}, \quad x \to \mathrm{Fil}(\mu_x)$$

is an embedding of X as an open complex submanifold of X, called the *Borel embedding*. Letting $K_h \subset G(\mathbb{R})$ denote the isotropy group of the point $h \in X$, it is well known that $X \cong G(\mathbb{R})/K_h$, and in fact $K_h = P(\mathbb{C}) \cap G(\mathbb{R})$. We have a commutative diagram

$$\begin{tikzcd} G(\mathbb{R})/K_h \arrow[r, hook] \arrow[d, "\cong"] & G(\mathbb{C})/P(\mathbb{C}) \arrow[d, "\cong"] \\ X \arrow[r, hook] & \check{X} \end{tikzcd}$$

Now, let $\mathcal{J}$ be a $G_{\mathbb{C}}$-equivariant vector bundle on $\check{X}$. This restricts to a $G(\mathbb{R})$-equivariant vector bundle $\beta^{-1}(\mathcal{J})$ on X. At this point we want to pass to the quotient to obtain a vector bundle on $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$, but this requires an extra condition. Let Z_s denote the largest subtorus of the center Z of G that splits over $\mathbb{R}$ but has no subtorus split over $\mathbb{Q}$, and let $G^c := G/Z_s$. Now, under the condition that the $G_{\mathbb{C}}$-action on $\mathcal{J}$ factors through $G^c_{\mathbb{C}}$, we can descend to the quotient and obtain the vector bundle

$$\mathcal{V}_K(\mathcal{J}) := G(\mathbb{Q})\backslash\beta^{-1}(\mathcal{J}) \times G(\mathbb{A}^\infty)/K$$

on $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ for sufficiently small K. We will also denote the sheaf of sections by $\mathcal{V}_K(\mathcal{J})$.

Remark 3.5. The condition that the $G_{\mathbb{C}}$-action factor through $G^c_{\mathbb{C}}$ ensures that after taking the quotient by Z_s, K can be made sufficiently small so that $K \cap Z(\mathbb{Q}) = \{1\}$. Restrict $\beta^{-1}(\mathcal{J})$ to the connected component X^+. Then the $G(\mathbb{R})$-action can be restricted to an action of $\Gamma'_g = gKg^{-1} \cap G(\mathbb{Q})_+$, and since $K \cap Z(\mathbb{Q}) = \{1\}$, we can identify Γ'_g with its image Γ_g in $G^{\mathrm{ad}}(\mathbb{R})^+$. We then define $\mathcal{V}_K(\mathcal{J})$ locally on $\Gamma_g \backslash X^+$ by taking the quotient. This condition can be avoided by instead considering stacks (see [Mil90] §III.8).

For any $G_{\mathbb{C}}$-equivariant vector bundle $\mathcal{J}$ on $\check{X}$, the parabolic P acts on the fiber $\mathcal{J}_\mu$, and the map $\mathcal{J} \to \mathcal{J}_\mu$ defines an equivalence of categories

$$\left\{G_{\mathbb{C}}\text{-equivariant vector bundles on } \check{X}\right\} \xrightarrow{\cong} \mathbf{Rep}_{\mathbb{C}}(P)$$

In particular, any representation ρ of the Levi subgroup $L \subset P$ gives a representation of P via the quotient $P \twoheadrightarrow L$ by the unipotent radical $R_u(P)$. Denote the resulting automorphic line bundle by $\mathcal{V}_K(\rho)$.

Now, suppose $\eta \in X^*_{+,L}(T)$ is an L-dominant character of the maximal torus T. We obtain an L-equivariant line bundle $\mathscr{L}(\eta)$ on $L/(B \cap L)$, and hence a representation $V_\eta = H^0(L/(B \cap L), \mathscr{L}(\eta))$ of L. Denote the resulting automorphic bundle by $\mathcal{V}_K(\eta)$.

Example 3.6. Let (G, X) be a Hodge type Shimura datum and let μ be the corresponding cocharacter. Then there exists a symplectic embedding $G \hookrightarrow \mathrm{GSp}(V, \psi)$, and let V_+ denote the maximal isotropic subspace of V on which μ acts by multiplication by $x \in \mathbb{G}_m$. Denote by L the Levi given by $L := \mathrm{Cent}_G(\mu)$; then L fixes V_+. Define the character

$$\eta_\omega : L \to \mathbb{G}_m, \quad g \to \det(g|_{V_+})^{-1}$$

The resulting automorphic bundle $\mathcal{V}_K(\eta)$ on $\mathrm{Sh}_K(G, X)$ is called the *Hodge bundle*, and can equivalently be given by pushing forward the canonical bundle on the universal abelian scheme $\mathscr{A} \to \mathrm{Sh}_K(G, X)$.

In [Mil90] §III.5, Milne shows that automorphic vector bundles have canonical models over the reflex field E. Milne provides a *standard principal bundle* $P_K(G, X)$ defined over E, which is a G^c-torsor over $\mathrm{Sh}_K(G, X)$ and comes equipped with a rational map $\gamma : P_K(G, X) \to \check{X}$ defined over E. Thus, any G^c-equivariant bundle $\mathcal{J}$ on $\check{X}$ which is defined over E can be pulled back to $P_K(G, X)$ and descended to $\mathrm{Sh}_K(G, X)$, giving a canonical model over E. Specifically, this gives a canonical model for $\mathcal{V}_K(\rho)$ for any representation ρ of L defined over E.

Integral canonical models of automorphic vector bundles were constructed by Lovering in [Lov17b]. Lovering extends Milne's standard principal bundle to a $G^c_{\mathcal{O}_{E,\mathfrak{p}}}$-torsor $\mathscr{P}_K$ over the integral canonical model $\mathscr{S}_K$. By considering $G^c_{\mathcal{O}_{E,\mathfrak{p}}}$-equivariant vector bundles $\mathcal{J}$ on the integral analogue of $\check{X}$, Lovering repeats Milne's construction to obtain integral canonical models for $\mathcal{V}_K(\mathcal{J})$ over $\mathcal{O}_{E,\mathfrak{p}}$. We can then reduce mod p to obtain automorphic bundles over the special fiber $\mathcal{S}_K$ of $\mathscr{S}_K$. For a more complete picture on the integral theory in the Hodge type case, see [GK19a] §4.1.9.

3.2 Hasse Invariants on Shimura Varieties

In this section, we introduce Hasse invariants on abelian type Shimura varieties. They are essentially pullbacks of the group theoretic Hasse invariants from §2.3 via the Ekedahl-Oort stratification map $\zeta_K : \mathcal{S}_K(G, X) \to G\text{-}\mathbf{Zip}^\mu$ defined in the next section.

3.2.1 Ekedahl-Oort Stratification and the Zip Period Map

The Ekedahl-Oort stratification of $\mathscr{A}_g \otimes \mathbb{F}_p$, the moduli space of principally polarized abelian varieites of dimension g over $\mathbb{F}_p$, was initially constructed by Oort in [Oor01]. Each stratum consists of those abelian varieties A over $\mathbb{F}_p$ whose p-torsion subgroups $A[p]$ belong to a specific isomorphism class. Oort showed that each isomorphism class is determined by combinatorial data associated to a filtration of $A[p]$, called *elementary sequences*. The fundamental property of this stratification is that passing to the boundary of a stratum corresponds to "degenerating the p-structure".

Later, Moonen ([Moo01]) generalized the stratification to Shimura varieties of PEL types A (unitary), C (symplectic), and D (orthogonal). He showed that Oort's elementary sequences can be obtained via certain partial flags in the Dieudonné module of $A[p]$. The partial flags have different structure depending on the PEL type: for example, symplectic flags are used in type C. Furthermore, Moonen showed that the set of strata is in bijection with a certain quotient of the Weyl group $W(G)$ of the group G appearing in the Shimura datum.

A general framework for these discrete invariants was given by Moonen and Wedhorn in [MW04]. They defined an *F-zip* over an $\mathbb{F}_q$-scheme S to be a tuple $\underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet)$ with

- M a locally free $\mathcal{O}_S$-module of finite rank
- $C^\bullet$ a descending filtration of M
- $D_\bullet$ an ascending filtration of M
- $\varphi_\bullet$ a collection of $\mathcal{O}_S$-linear isomorphisms

  $$\varphi_n : (\mathrm{gr}^n_C)^{(q)} \xrightarrow{\sim} \mathrm{gr}^D_n$$

  for $n \in \mathbb{Z}$

For each point $s \in S$, the filtration $C^\bullet$ is assigned a *type* $\tau^s_{C^\bullet} : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ given by $m \to \dim_{k(s)}(\mathrm{gr}^m_{C\otimes k(s)})$. The *type* of the F-zip is then given by $\tau : s \to \tau^s_{C^\bullet}$. When S is connected, τ is simply a function $\mathbb{Z} \to \mathbb{Z}_{\geq 0}$ with finite support. Moonen and Wedhorn also consider "F-zips with additional structure" such as symplectic (resp. orthogonal) F-zips, where the filtrations are symplectic (resp. symmetric) flags.

Example 3.7. The F-zip associated to an abelian variety A over a perfect field κ is given by its algebraic de Rham cohomology $H^1_{\mathrm{dR}}(A/\kappa)$ along with the Hodge and conjugate filtrations and the Cartier isomorphisms which relate these two filtrations; this serves as a blueprint for the definition of F-zip. In this case, the polarization of A gives a symplectic structure to the F-zip.

For each $\mathbb{F}_q$-scheme S, we define the category of F-zips over S as $F\text{-}\mathbf{Zip}_S$. For a morphism of $\mathbb{F}_q$-schemes $T \to S$, there is a pullback $F\text{-}\mathbf{Zip}_S \to F\text{-}\mathbf{Zip}_T$, and hence we obtain a stack $F\text{-}\mathbf{Zip}$ fibered over the category of $\mathbb{F}_q$-schemes. For each τ, we obtain a substack $F\text{-}\mathbf{Zip}^\tau$ of F-zips of type τ.

Now, let k be an algebraically closed field of characteristic p, and let V be a finite dimensional $\mathbb{F}_p$-vector space. In the symplectic (resp. orthogonal) case, let ψ denote a symplectic (resp. symmetric) perfect bilinear pairing on V. Let G denote $\mathrm{GL}(V)$ in the standard case, $\mathrm{Sp}(V,\psi)$ in the symplectic case, and $\mathrm{SO}(V,\psi)$ in the orthogonal case. A type τ determines a parabolic subgroup of G, given by the parabolic which fixes any (standard, symplectic, symmetric) filtration $C^\bullet$ of type τ. With notation as in §2.2.1, let $I \subset \Delta$ be the type of this parabolic and ${}^IW = W_I\backslash W$ the Weyl group quotient. Moonen and Wedhorn give a bijection (Theorem 4.4 and Corollary 5.4 in [MW04])

$$\left\{ \begin{array}{c} \text{isomorphism classes of (standard, symplectic,} \\ \text{orthogonal) } F\text{-zips of type } \tau \text{ over } k \end{array} \right\} \xrightarrow{\sim} {}^IW$$

They then show that an F-zip $\underline{M}$ of type τ over an $\mathbb{F}_q$-scheme S determines a stratification of S: the existence of $\underline{M}$ is equivalent to a morphism of stacks

$$\zeta : S \to F\text{-}\mathbf{Zip}^\tau$$

and the strata of S are the fibers of this morphism, indexed by ${}^IW$. In the case of a mod p PEL Shimura variety $\mathcal{S}$, the universal abelian scheme $\mathcal{A} \to \mathcal{S}$ defines a universal F-zip over $\mathcal{S}$ (which is symplectic for PEL type C, etc.), and the resulting stratification is precisely the Ekedahl-Oort stratification defined above. Equivalently, the morphism $\zeta : \mathcal{S} \to F\text{-}\mathbf{Zip}^\tau$ is obtained by mapping an abelian scheme to its F-zip.

For all the successes of [MW04], the authors were unable to find a framework that works for an arbitrary reductive group G. This was later provided by Pink, Wedhorn, and Ziegler ([PWZ11]) via the theory of *G-zips*, as defined in §2.1.4. The Ekedahl-Oort stratification for a general Shimura variety is thus given in the form of a morphism of stacks

$$\zeta : \mathcal{S}_K(G,X) \to G\text{-}\mathbf{Zip}^\mu$$

where the strata are again indexed by ${}^IW$ in light of equation (2.1).

For Hodge type Shimura varieties, the morphism ζ was constructed by Zhang in [Zha18]. For abelian type Shimura varieties, ζ should take the form

$$\zeta : \mathcal{S}_K(G,X) \to G^c\text{-}\mathbf{Zip}^{\mu^c}$$

where $G^c := G/Z_s$ is the quotient of G by the largest $\mathbb{R}$-split $\mathbb{Q}$-anisotropic subtorus Z_s (as in §3.1.3) and μ^c is given by composing μ with the quotient map. This morphism was first constructed by Shen and Zhang in [SZ22], who give an initial construction using the adjoint group of G, but give a more complete construction using Lovering's crystalline canonical model ([Lov17a]). A more refined construction was provided by Imai, Kato, and Youcis ([IKY23]) using their prismatic realization functor, which serves as an upgrade to Lovering's crystalline model.

3.2.2 Hasse Invariants and the Griffiths Bundle

Let (G, X) be a proper abelian type Shimura datum such that $G = G^c$ (see §3.1.3) and let $\mathscr{S}_K(G, X)$ denote the integral canonical model over $\mathcal{O}_{E,\mathfrak{p}}$ of the corresponding abelian type Shimura variety, where $K = K^pK_p$ with $K_p = G(\mathbb{Z}_p)$ and $K^p \subset G(\mathbb{A}^{\infty,p})$. Write $\mathcal{S}_K(G, X)$ for the reduction mod p of $\mathscr{S}_K(G, X)$. Let

$$\zeta_K : \mathcal{S}_K(G, X) \to G\text{-}\mathbf{Zip}^\mu$$

denote the Ekedahl-Oort stratification as in §3.2.1, where μ is the associated cocharacter. With notation as in §3.2.1, for each $w \in {}^IW$ we write $\mathcal{S}_w := \zeta^*[E_\mathcal{Z}\backslash G_w]$ (resp. $\overline{\mathcal{S}}_w := \zeta_K^*[E_\mathcal{Z}\backslash \overline{G}_w]$) for the corresponding EO-stratum (resp. closed EO-stratum). By construction, the EO-stratification is $G(\mathbb{A}^{\infty,p})$-equivariant, i.e. given $K'^p \subset K^p$ and $g \in G(\mathbb{A}^{\infty,p})$, it satisfies the following commutative triangles:

$$\begin{tikzcd} \mathcal{S}_{K'} \arrow[d, "\pi_{K'/K}"'] \arrow[dr, "\zeta_{K'}"] & \\ \mathcal{S}_K \arrow[r, "\zeta_K"'] & G\text{-}\mathbf{Zip}^\mu \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{S}_K \arrow[d, "g"'] \arrow[dr, "\zeta_K"] & \\ \mathcal{S}_{g^{-1}Kg} \arrow[r, "\zeta_{g^{-1}Kg}"'] & G\text{-}\mathbf{Zip}^\mu \end{tikzcd}$$

Fix a homomorphism $r : G \to \mathrm{GL}(V)$ with central kernel. Let $\mathscr{G}$ denote the Griffiths line bundle on $G\text{-}\mathbf{Zip}^\mu$, i.e. $\mathscr{G} = \mathscr{L}_{\mathrm{Zip}}(\mathrm{grif}(G, \mu, r))$. By Proposition 2.21, the pullback $\zeta_K^*\mathscr{G}$ is an ample line bundle on $\mathcal{S}_K(G, X)$. As we saw in Example 2.19, when (G, X) is Hodge type, $\zeta_K^*\mathscr{G}$ is precisely the Hodge bundle, so we will simply write ω for this line bundle in the general abelian type case as well. By Corollary 2.20, some power $\mathscr{G}^N$ of $\mathscr{G}$ admits group theoretic Hasse invariants $h_w \in H^0([E_\mathcal{Z}\backslash\overline{G}_w], \mathscr{G}^N)$ whose non-vanishing locus is $[E_\mathcal{Z}\backslash G_w]$.

Remark 3.8. The dependence of this setup on the homomorphism $r : G \to \mathrm{GL}(V)$ in the definition of the Griffiths character is minimal: a different choice of r will result in the same line bundle ω, with a potentially different Hecke action by $G(\mathbb{A}^{\infty,p})$. Thus, we will often drop r from the notation.

Proposition 3.9. *For every EO stratum $\mathcal{S}_w \subset \mathcal{S}_K$, the section $\zeta_K^*h_w \in H^0(\overline{\mathcal{S}}_w, \omega^N)$ is $G(\mathbb{A}^{\infty,p})$-equivariant and has non-vanishing locus $\mathcal{S}_w$.*

Proof. Since the EO stratification is $G(\mathbb{A}^{\infty,p})$-equivariant, so are the sections $\zeta_K^*h_w$. Since the section h_w has non-vanishing locus $[E_\mathcal{Z}\backslash G_w]$ and $\mathcal{S}_w = \zeta_K^*[E_\mathcal{Z}\backslash G_w]$, the proof is complete. □

We now make precise the compatibility of ω with the classical Hodge bundle. Let $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ denote the adjoint Shimura datum corresponding to (G, X), and let K^{ad} be the image of K in $G^{\mathrm{ad}}(\mathbb{A}^\infty)$. The adjoint map $G \to G^{\mathrm{ad}}$ induces a morphism of Shimura varieties

$$\mathcal{S}_K(G, X) \to \mathcal{S}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$$

Since the underlying topological space of $G\text{-}\mathbf{Zip}^\mu$ is in bijection with ${}^IW$ (see §2.2.2), we have a bijection between the underlying topological spaces of $G\text{-}\mathbf{Zip}^\mu$ and $G^{\mathrm{ad}}\text{-}\mathbf{Zip}^{\mu^{\mathrm{ad}}}$. Since the morphism of zip data $(G, \mu) \to (G^{\mathrm{ad}}, \mu^{\mathrm{ad}})$ induces a morphism of stacks $G\text{-}\mathbf{Zip}^\mu \to G^{\mathrm{ad}}\text{-}\mathbf{Zip}^{\mu^{\mathrm{ad}}}$, we obtain a commutative diagram

$$
\begin{tikzcd}
\mathcal{S}_K(G,X) \arrow[r, "\zeta"] \arrow[d, "\pi^{\mathrm{ad}}"'] & G\text{-}\mathbf{Zip}^{\mu} \arrow[d, "\tilde{\pi}^{\mathrm{ad}}"] \\
\mathcal{S}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}},X^{\mathrm{ad}}) \arrow[r, "\zeta^{\mathrm{ad}}"'] & G^{\mathrm{ad}}\text{-}\mathbf{Zip}^{\mu^{\mathrm{ad}}}
\end{tikzcd}
$$

Proposition 3.10. *Using the above notation, let ω (resp. ω^{ad}) denote the Griffiths bundle on $\mathcal{S}_K(G,X)$ (resp. $\mathcal{S}_{K^{ad}}(G^{ad},X^{ad})$). Then $\omega \cong (\pi^{ad})^*\omega^{ad}$.*

Proof. Denote by $s : \tilde{G} \to G$ the map from §2.3.2, and similarly for $s^{\mathrm{ad}} : \tilde{G} \to G^{\mathrm{ad}}$. These satisfy $\mathrm{ad} \circ s = s^{\mathrm{ad}}$. By Proposition 2.18 (iii), the Griffiths characters satisfy

$$s^* \operatorname{grif}(G,\mu,r) = -\frac{(\alpha,\alpha)}{4} S(\alpha^\vee, r)\mu^{\mathrm{ad}} = (s^{\mathrm{ad}})^* \operatorname{grif}(G^{\mathrm{ad}},\mu^{\mathrm{ad}},r^{\mathrm{ad}})$$

as characters in $X^*(\tilde{T})_{\mathbb{Q}}$ and hence

$$s^* \operatorname{grif}(G,\mu,r) = s^*\big(\operatorname{ad}^* \operatorname{grif}(G^{\mathrm{ad}},\mu^{\mathrm{ad}},r^{\mathrm{ad}})\big)$$

Now, since $s^* : X^*(T)_{\mathbb{Q}} \to X^*(\tilde{T})_{\mathbb{Q}}$ is a surjection, we see that

$$\operatorname{grif}(G,\mu,r) = \operatorname{ad}^* \operatorname{grif}(G^{\mathrm{ad}},\mu^{\mathrm{ad}},r^{\mathrm{ad}}) + \chi$$

where $\chi \in X^*(T)_{\mathbb{Q}}$ is a character such that $s^*\chi$ is trivial. On the level of Shimura varieties, this implies that

$$\omega \cong (\pi^{\mathrm{ad}})^*\omega^{\mathrm{ad}} \otimes \mathcal{V}(\chi)$$

However, since $s^*\chi$ is trivial, the line bundle $\mathcal{V}(\chi)$ is trivial on $\mathcal{S}_K(G,X)$, and thus the result follows. ∎

Now, suppose (G,X) is an abelian type Shimura datum, and let (G_1,X_1) be an associated Hodge type Shimura datum, i.e. there exists a central isogeny $\varphi : G_1^{\mathrm{der}} \to G^{\mathrm{der}}$ which induces an isomorphism of adjoint Shimura data $\varphi : (G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) \cong (G^{\mathrm{ad}}, X^{\mathrm{ad}})$. We may compatibly choose level subgroups $K \subset G(\mathbb{A}^{\infty,p}), K_1 \subset G_1(\mathbb{A}^{\infty,p})$ so that $K^{\mathrm{ad}} = K_1^{\mathrm{ad}}$, where K^{ad} is the image of K under the adjoint map (similarly for K_1). This gives a commutative diagram

$$
\begin{tikzcd}
\mathcal{S}_{K_1}(G_1,X_1) \arrow[dr, "\pi_1^{\mathrm{ad}}"] & \\
& \mathcal{S}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}},X^{\mathrm{ad}}) \\
\mathcal{S}_K(G,X) \arrow[ur, "\pi^{\mathrm{ad}}"'] &
\end{tikzcd}
$$

We obtain the following easy corollary:

Corollary 3.11. *Using the above notation, the Hodge bundle ω_1 on $\mathcal{S}_{K_1}(G_1,X_1)$ is compatible with the Griffiths bundle ω on $\mathcal{S}_K(G,X)$ in the following sense: for ω^{ad} the Griffiths bundle on $\mathcal{S}_{K^{ad}}(G^{ad},X^{ad})$, we have*

$$\omega_1 \cong (\pi_1^{ad})^*\omega^{ad} \qquad \textit{and} \qquad \omega \cong (\pi^{ad})^*\omega^{ad}$$

Proof. As we saw in Example 2.19, the Griffiths bundle of a Hodge type Shimura datum is precisely the Hodge bundle. Thus, the result easily follows from Proposition 3.10. ∎

3.2.3 Length Stratification and Length Hasse Invariants

Recalling the notation of §2.2.1, let $d := \ell(w_{0,I}w_0)$. The length of any $w \in {}^I W$ satisfies $0 \leq \ell(w) \leq d$. For any $0 \leq j \leq d$, define

$$G_j = \bigcup_{\ell(w)=j} G_w \quad \text{and} \quad \overline{G}_j = \bigcup_{\ell(w)\leq j} G_w$$

endowed with the reduced subscheme structure. We call G_j the jth *length stratum* of G. By [GK19a] Lemma 5.2.1, we have $\overline{G}_j = \bigcup_{\ell(w)=j} \overline{G}_w$. Moreover, the Griffiths bundle admits characteristic sections on the length strata, called *group theoretic length Hasse invariants*:

Proposition 3.12. *([GK19a] Proposition 5.2.2) If χ is a Hasse generator, then there exists an integer N and for each $0 \leq j \leq d$ a section $h_j \in H^0([E\backslash\overline{G}_j], \mathcal{V}(N\chi))$ such that the nonvanishing locus of h_j is exactly $[E\backslash G_j]$.*

Let $\mathcal{S}_K(G,X)$ be the reduction mod p of a proper abelian type Shimura variety, and let $\zeta_K : \mathcal{S}_K(G,X) \to G\text{-}\mathbf{Zip}^\mu$ be the zip period map. Just as with the E-O stratification, we can define the *length stratification* by

$$\mathcal{S}_j = \bigsqcup_{\ell(w)=j} \mathcal{S}_w \quad \text{and} \quad \overline{\mathcal{S}}_j = \bigsqcup_{\ell(w)\leq j} \mathcal{S}_w$$

We have $\overline{\mathcal{S}}_j = \bigcup_{\ell(w)=j} \overline{\mathcal{S}}_w$, and pulling back the group theoretic length Hasse invariants by ζ_K gives *length Hasse invariants* $\zeta_K^* h_j$. By abuse of notation, we will often write h_j for $\zeta_K^* h_j$. A key property of the length Hasse invariants is that they are *injective*.

Definition 3.13. Let X be a scheme and $\mathcal{L}$ a line bundle on X. A section $s \in H^0(X, \mathcal{L})$ is called *injective* (or *regular*) if the map of sheaves $\mathcal{O}_X \to \mathcal{L}$ given by multiplication by s is injective.

Proposition 3.14. *Given a proper abelian type Shimura datum (G,X), the following hold:*

(i) The length Hasse invariant $h_j \in H^0(\overline{\mathcal{S}}_j, \omega^{N_j})$ is $G(\mathbb{A}^{\infty,p})$-equivariant and has nonvanishing locus $\mathcal{S}_j$.

(ii) The length strata $\mathcal{S}_j$ and $\overline{\mathcal{S}}_j$ have dimension j.

(iii) $\mathcal{S}_j$ is open dense in $\overline{\mathcal{S}}_j$. Equivalently, h_j is injective.

Proof. These claims follow immediately from [GK19a] Corollary 6.4.4. □

3.2.4 Hasse Regular Sequences

Write $\mathcal{S}_K^n(G,X)$ for the reduction mod p^n of $\mathscr{S}_K(G,X)$, so $\mathcal{S}_K^1(G,X) = \mathcal{S}_K(G,X)$ under our notation. Let $(\mathcal{S}_K^n)_{K^p}$ denote the tower of mod p^n Shimura varieties over all $K^p \subset G(\mathbb{A}^{\infty,p})$.

Definition 3.15. A *Hasse regular sequence of length r* in $(\mathcal{S}_K^n)_{K^p}$ is the data, for each K^p, of a sequence $(Z_j, a_j, f_j)_{j=0}^r$ satisfying

(i) One has $Z_0 = \mathcal{S}^n_K$ and $(a_j)_{j=0}^r$ is a sequence of positive integers independent of K^p.

(ii) For all $j > 0$, $Z_j \subset Z_{j-1}$ is a closed subscheme with $(Z_j)_{\text{red}} = \overline{\mathcal{S}}_{d-j}$, the $(d-j)$th length stratum (with d as in §3.2.3).

(iii) For all j, $f_j \in H^0(Z_j, \omega^{a_j})$ is a section which Zariski locally restricts to a power of the length Hasse invariant h_{d-j} on $\overline{\mathcal{S}}_{d-j}$.

(iv) For all $j < r$, the zero scheme of f_j is Z_{j+1}.

Remark 3.16. (i) The existence (and uniqueness) of the section f_j is given by [GK19a] Theorem 5.1.1. Furthermore, since the length Hasse invariant h_{d-j} is injective by Proposition 3.14 (iii), f_j is also injective by [GK19a] Corollary 7.2.5.

(ii) Since h_{d-j} is $G(\mathbb{A}^{\infty,p})$-equivariant, so is Z_j and f_j.

(iii) By 3.14 (ii), Z_j has dimension $d - j$.

Definition 3.17. A system of subschemes $Z \subset \mathcal{S}^n_K$ is *Hasse regular* if $Z = Z_r$ for some Hasse regular sequence $(Z_j, a_j, f_j)_{j=0}^r$.

4 Applications to Galois Representations

4.1 The Langlands Correspondence

We give a brief overview of the role of Shimura varieties in the Langlands correspondence. For a more detailed introduction, see [GH24] §15.

4.1.1 Automorphic Representations

Let G be a reductive group over $\mathbb{Q}$. For simplicity, we will assume that G^{der} is $\mathbb{Q}$-anisotropic (thus if G is part of a Shimura datum (G, X), the resulting Shimura variety $\text{Sh}_K(G, X)$ is proper by Theorem 3.4). Let $K^{\max}_\infty \subset G(\mathbb{R})$ be a maximal compact subgroup, and let K_∞ be a compact subgroup such that

$$(K^{\max}_\infty)^\circ \subseteq K_\infty \subseteq K^{\max}_\infty$$

Denote by A_G the maximal $\mathbb{Q}$-split subtorus of $Z(G)$, and write $[G] := G(\mathbb{Q})\backslash G(\mathbb{A})/A_G(\mathbb{R})^+$ for the adelic quotient of G. Note that setting $X := A_G(\mathbb{R})^+\backslash G(\mathbb{R})/K_\infty$ gives a Shimura datum (G, X).

Automorphic representations are certain representations of the adelic points $G(\mathbb{A})$ of G which appear as subrepresentations of certain L^2 spaces. Since we are assuming that G^{der} is $\mathbb{Q}$-anisotropic, we can simplify the definitions. For a character $\chi : A_G(\mathbb{R})^+ \to \mathbb{C}^\times$, define the representation $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \chi)$ of $G(\mathbb{A})$ to be the space of functions $f : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ such that

(i) $f(zg) = \chi(z)f(g)$ for all $z \in A_G(\mathbb{R})^+$

(ii) $f\chi^{-1} : [G] \to \mathbb{C}$ is measurable

(iii) $\int_{[G]} |(f\chi^{-1})(g)|^2 \, dg < \infty$

where the action of $G(\mathbb{A})$ is given by $(g \cdot f)(x) := f(xg)$. By [GH24] Chapter 9, this space has a Hilbert space decomposition

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \chi) = \widehat{\bigoplus}_{\pi} m(\pi)\pi$$

where π runs over the irreducible unitary subrepresentations of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \chi)$ and $m(\pi)$ is an integer. An *automorphic representation* of G is an irreducible unitary representation π of $G(\mathbb{A})$ occurring in the above decomposition.

Let $\mathfrak{g}$ denote the Lie algebra of $G(\mathbb{R})$. By a well-known theorem of Flath, an automorphic representation π of G has an associated tensor product

$$\pi_\infty \otimes \pi^\infty$$

where π_∞ is an irreducible $(\mathfrak{g}, K_\infty^{\max})$-module for $G(\mathbb{R})$ (the "archimedean component") and π^∞ is an irreducible smooth representation of $G(\mathbb{A}^\infty)$ (the "non-archimedean component"). The choice of a smooth reductive model of G over $\mathbb{Z}[1/N]$ determines a restricted tensor product decomposition

$$\pi^\infty = \bigotimes_p{}' \pi_p$$

where π_p is an irreducible smooth representation of $G(\mathbb{Q}_p)$, corresponding to the decomposition $G(\mathbb{A}^\infty) = \Pi_p' G(\mathbb{Q}_p)$.

4.1.2 Langlands Correspondence

Fix a Borel pair (B, T) for G. Write $\rho \in X^*(T)_\mathbb{C}$ for the half sum of the positive roots of G. Following Buzzard-Gee ([BG14]), a character $\chi \in X^*(T)_\mathbb{C}$ is called *L-algebraic* (resp. *C-algebraic*) if $\chi \in X^*(T)$ (resp. $\chi \in X^*(T) + \rho$). For an irreducible $(\mathfrak{g}, K_\infty^{\max})$-module π_∞, write χ_∞ for its infinitessimal character, which is identified with an element of $X^*(T)_\mathbb{C}/W$ via the Harish-Chandra isomorphism. Now, if π is an automorphic representation of G, we call it *L-algebraic* (resp. *C-algebraic*) if the infinitessimal character χ_∞ of the archimedean component π_∞ is L-algebraic (resp. C-algebraic).

Associated to the group G is its *Langlands dual group* ${}^L G$. To define it, first consider the root datum $(X^*, \Phi, X_*, \Phi^\vee)$ of G consisting of the character lattice, roots, cocharacter lattice, and coroots, respectively. There exists a group $\widehat{G}$ with root datum $(X_*, \Phi^\vee, X^*, \Phi)$ given by swapping the characters with cocharacters and roots with coroots. Define the Langlands dual group to be

$${}^L G := \widehat{G} \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

Fix an isomorphism of abstract fields $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. For the purposes of this paper, the primary conjecture of the Langlands correspondence is the following: given a connected reductive group G over $\mathbb{Q}$ and an L-algebraic automorphic representation π of G, for every prime p there exists an associated continuous Galois representation

$$R_{p,\iota}(\pi) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to {}^L G(\overline{\mathbb{Q}}_p)$$

Most progress in establishing this association has been limited to the following weaker construction. Given a representation $r : {}^L G \to \mathrm{GL}_m$ for some $m \geq 1$ which factors through $\hat{G} \rtimes \mathrm{Gal}(F/\mathbb{Q})$ for some finite Galois extension F over which G splits, it can be simpler to construct the representation

$$r \circ R_{p,\iota}(\pi) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_m$$

4.1.3 Hecke Algebras

Let R be a ring (typically $\mathbb{C}$, although we will consider $\mathbb{Z}_p$ as well). The *Hecke algebra* $\mathcal{H}(G; R)$ of $G(\mathbb{A}^\infty)$ with coefficients in R is the set of locally constant compactly supported functions $f : G(\mathbb{A}^\infty) \to R$ with the convolution operator

$$(f_1 * f_2)(g) := \int_{G(\mathbb{Q}_v)} f_1(gh^{-1}) f_2(h)\, dh$$

If $K \subset G(\mathbb{A}^\infty)$ is a compact open subgroup, we denote by $\mathcal{H}(G, K; R)$ the subalgebra of $\mathcal{H}(G; R)$ consisting of K-bi-invariant functions. The identity element of $\mathcal{H}(G, K; R)$ is given by $e_K = \frac{1}{\mathrm{Vol}(K)} \mathbb{1}_K$, where $\mathbb{1}_K$ is the characteristic function of K.

Remark 4.1. We are suppressing the notion of "Haar measure" which is necessary in these definitions. See [GH24] Chapter 3 for more details.

The importance of Hecke algebras in the Langlands philosophy is given by the following fact: given a characteristic 0 field F, there exists an equivalence of categories

$$\left\{ \begin{array}{c} \text{smooth } F\text{-representations} \\ \text{of } G(\mathbb{A}^\infty) \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{non-degenerate} \\ \mathcal{H}(G; F)\text{-representations} \end{array} \right\}$$

where a smooth F-representation V of $G(\mathbb{A}^\infty)$ can be given the structure of $\mathcal{H}(G; F)$-representation by

$$f \cdot v := \int_{G(\mathbb{A}^\infty)} f(h) hv\, dh$$

Thus, for each compact open $K \subset G(\mathbb{A}^\infty)$, we can consider the $\mathcal{H}(G, K; F)$-representation V^K. In fact, $V^K = e_K V$ and $\mathcal{H}(G, K; F) = e_K * \mathcal{H}(G; F) * e_K$. We call a representation V of G *admissible* if V^K is finite-dimensional for all open compact $K \subset G(\mathbb{A}^\infty)$.

The study of global Hecke algebras can be reduced to the study of local Hecke algebras. For a non-archimedean place v, the *local Hecke algebra* $\mathcal{H}_v(G_v; R)$ of $G(\mathbb{Q}_v)$ with coefficients in R is the set of locally constant compactly supported functions $f : G(\mathbb{Q}_v) \to R$ under the convolution operator as given above. When G is unramified at v, there exists a hyperspecial subgroup $K_v \subset G(\mathbb{Q}_v)$, and we define the *unramified Hecke algebra* $\mathcal{H}_v(G_v, K_v; R)$ of $G(\mathbb{Q}_v)$ as the subalgebra of $\mathcal{H}_v(G_v; R)$ consisting of K_v-bi-invariant functions. An important theorem of Flath states that an irreducible admissible $\mathcal{H}(G; R)$-module W can be factored into admissible $\mathcal{H}_v(G_v; R)$-modules

$$W \cong \bigotimes_{v \nmid \infty}{}' W_v$$

where $\dim_R W_v^{K_v} = 1$ for all but finitely many v.

With regard to the Langlands correspondence, the role of Hecke algebras can be summarized with the following diagram:

$$\begin{tikzcd}
\left\{\begin{matrix}\text{automorphic}\\ \text{representations of } G\end{matrix}\right\} \arrow[r, dashed, "R_{p,\iota}"] \arrow[d, "\pi\to\pi^\infty"'] & \left\{\begin{matrix}\text{Galois representations}\\ \text{valued in } {}^LG(\overline{\mathbb{Q}}_p)\end{matrix}\right\} \\
\left\{\begin{matrix}\text{smooth admissible}\\ \text{representations of}\\ G(\mathbb{A}^\infty)\end{matrix}\right\} \arrow[r, "\sim"] & \left\{\begin{matrix}\text{non-degenerate}\\ \mathcal{H}(G)\text{-representations}\end{matrix}\right\} \arrow[u, dashed]
\end{tikzcd}$$

That is, if we can associate a Galois representation to an $\mathcal{H}(G)$-representation, we will have established $R_{p,\iota}$. This of course does not make the problem much easier, but it does moderately expand our toolset.

4.1.4 Torsion and Pseudo-representations

In the famous proof of Fermat's Last Theorem ([Wil95], [TW95]), certain *modularity lifting* techniques were crucial to the argument. These techniques give a way to prove that Galois representations coming from geometry are in fact automorphic, an important step in establishing the Langlands correspondence. The basic idea is this: given a geometric Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_m(\overline{\mathbb{Q}}_p)$ with residual representation $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_m(\overline{\mathbb{F}}_p)$, if $\overline{\rho}$ is automorphic (i.e. corresponds to the residual representation of an automorphic representation π of GL_m) and we have a few mild hypotheses on $\rho, \overline{\rho}, \pi$, then one hopes to show that ρ is automorphic.

The original modularity lifting method of Taylor and Wiles was limited to cases when the automorphic representations arose from the middle degree cohomology of Shimura varieties (i.e. "regular weight"). Calegari and Geraghty ([CG18]) extended this method to cover the non-regular cases, but the proofs depend on the conjectured existence of certain *torsion analogues* of the Langlands correspondence:

$$\begin{tikzcd}
\left\{\begin{matrix}\text{automorphic}\\ \text{representations}\end{matrix}\right\} \arrow[r, dashed] \arrow[d, dashed, leftrightarrow, "\text{torsion analogue}"] & \left\{\begin{matrix}\text{Galois}\\ \text{representations}\end{matrix}\right\} \arrow[d, dashed, leftrightarrow, "\text{torsion analogue}"] \\
\left\{\begin{matrix}\text{systems of Hecke eigenvalues appearing in}\\ \text{the cohomology of a locally symmetric}\\ \text{space with mod } p^n \text{ coefficients}\end{matrix}\right\} \arrow[r, dashed] & \left\{\begin{matrix}\text{mod } p^n\text{-valued Galois}\\ \text{pseudo-}\\ \text{representations}\end{matrix}\right\}
\end{tikzcd}$$

We now explain these torsion analogues more precisely, in the context of the coherent cohomology of Shimura varieties. Let (G, X) be a proper abelian type Shimura datum. Fix a prime $p \notin \mathrm{Ram}(G)$. For every $v \notin \mathrm{Ram}(G) \cup \{p\}$, let K_v be a hyperspecial subgroup of $G(\mathbb{Q}_v)$. Let $\mathcal{H}_v = \mathcal{H}_v(G_v, K_v; \mathbb{Z}_p)$ be the unramified Hecke algebra of G at v, with $\mathbb{Z}_p$ coefficients, normalized by the unique Haar measure which assigns K_v volume 1. Define the unramified global Hecke algebra (away from p) by

$$\mathcal{H} := \sideset{}{'}\bigotimes_{v \notin \mathrm{Ram}(G) \cup \{p\}} \mathcal{H}_v$$

We recall the Hecke action on the cohomology of $\mathcal{S}^n_K$. Write $K_g = K \cap gKg^{-1}$. Then $g^{-1}K_g g = K \cap g^{-1}Kg$. With notation as in §3.1.1, let $\pi_1 := \pi_{K_g/K}$ and $\pi_2 = \pi_{g^{-1}K_g g/K} \circ g$. Then the $\pi_j : \mathcal{S}^n_{K_g} \to \mathcal{S}^n_K$ are finite étale projections, and hence there is a trace map

$$\operatorname{tr}\pi_j : \pi_{j,*}\mathcal{O}_{\mathcal{S}^n_{K_g}} \to \mathcal{O}_{\mathcal{S}^n_K}$$

Let $\mathscr{F}$ be a $G(\mathbb{A}^{\infty,p})$-equivariant coherent sheaf on the tower $(\mathcal{S}^n_K)_{K^p}$. Then $\pi_1^*\mathscr{F} = \pi_2^*\mathscr{F}$, and the Hecke operator

$$T_g : H^i(\mathcal{S}^n_K, \mathscr{F}) \to H^i(\mathcal{S}^n_K, \mathscr{F})$$

is given by $T_g := \operatorname{tr}\pi_2 \circ \pi_1^*$.

Let $\eta \in X^*_{+,L}(T)$. Write $\mathcal{H}^{i,n}(\eta)$ for the image of $\mathcal{H}$ in $\operatorname{End}(H^i(\mathcal{S}^n_K, \mathcal{V}(\eta)))$. Similarly, write $\mathcal{H}^i(\eta)$ for the image of $\mathcal{H}$ in $\operatorname{End}(H^i(\mathscr{S}_K, \mathcal{V}(\eta)))$, with the action defined analogously as above. If Z is a Hasse regular subscheme of $\mathcal{S}^n_K$, by [GK19a] §8.1.6, the Hecke action on $\mathcal{S}^n_K$ restricts to Z, and we write $\mathcal{H}^{i,n}_Z(\eta)$ for the image of $\mathcal{H}$ in $\operatorname{End}(H^i(Z, \mathcal{V}(\eta)))$. The torsion analogue of the Langlands correspondence aims to attach to $\mathcal{H}^{i,n}(\eta)$ a Galois pseudo-representation

$$\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathcal{H}^{i,n}(\eta)$$

with compatibility between Frobenius maps and Hecke operators described in the next section.

4.1.5 Local-Global Compatibility of (Pseudo-)representations

We follow [GK19a] §10.2 in the following definitions. Recall from §4.1.2 the dual group $\widehat{G}$ and L-group ${}^LG$ of G. Write $\widehat{G}_v$ for the model over $\mathbb{Q}_v$ and ${}^LG_v := \widehat{G}_v \rtimes \operatorname{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$. For a representation $r : {}^LG \to \mathrm{GL}_m$, denote by $r_v : {}^LG_v \to \mathrm{GL}_m$ the restriction of r. Let $R_{\mathrm{fd}}({}^LG_v)$ denote the subalgebra of $\mathbb{Z}[{}^LG_v]$ generated by the characters of representations of ${}^LG_v$ just defined. Define $R^{\mathrm{ss}}_{\mathrm{fd}}({}^LG_v)$ to be the algebra obtained by viewing the elements in $R_{\mathrm{fd}}({}^LG_v)$ as functions on the set of $\widehat{G}_v(\mathbb{C})$-conjugacy classes in the coset $\widehat{G}_v \rtimes \mathrm{Frob}_v$, and then restricting to the subset of semisimple conjugacy classes.

Following the discussion in [GK19a] §10.2, the Satake isomorphism gives a canonical identification

$$\mathcal{H}_v[\sqrt{v}] \xrightarrow{\sim} R^{\mathrm{ss}}_{\mathrm{fd}}({}^LG_v)$$

which induces a canonical bijection

$$\left\{\begin{matrix}\text{unramified admissible}\\ \text{representations of } G(\mathbb{Q}_v)\end{matrix}\right\} \xleftarrow{\sim} \left\{\begin{matrix}\text{characters of}\\ \mathcal{H}_{v,\mathbb{C}}\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\widehat{G}_v(\mathbb{C})\text{-conjugacy classes}\\ \text{of semisimple elements}\\ \text{in } \widehat{G}_v \rtimes \mathrm{Frob}_v\end{matrix}\right\} \tag{4.1}$$

Given an unramified, admissible representation π_v of $G(\mathbb{Q}_v)$, write $\operatorname{Class}(\pi_v)$ for the corresponding $\widehat{G}_v(\mathbb{C})$-conjugacy class. For a representation $r : {}^LG \to \mathrm{GL}_m$, write $\operatorname{Class}(\pi_v, r_v)$ for the conjugacy class in $\mathrm{GL}_m(\mathbb{C})$ generated by the image $r(\operatorname{Class}(\pi_v))$. Using the isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$, write $\operatorname{Class}_{p,\iota}(\pi_v, r_v)$ for the corresponding conjugacy class in $\mathrm{GL}_m(\overline{\mathbb{Q}}_p)$.

Now, for each $j \geq 1$, denote the function

$${}^LG_v(\mathbb{C}) \to \mathbb{C}, \quad \tilde{g} \to \operatorname{tr}(r(\tilde{g})^j)$$

by $\mathrm{tr}^j(r)$. This function lies in $R^{\mathrm{ss}}_{\mathrm{fd}}({}^L G_v)$, and we denote its corresponding element in $\mathcal{H}_v[\sqrt{v}]$ (via the above isomorphism) by $T_v^{(j)}(r)$. Write $T_v^{(j)}(r; i, n, \eta)$ for the image of $T_v^{(j)}(r)$ in $\mathcal{H}^{i,n}(\eta)$. If $\mathcal{H} \to \mathcal{H}^{i,n}(\eta)$ factors through $\mathcal{H}^{i',n'}(\eta')$, then $T_v^{(j)}(r; i', n', \eta')$ maps to $T_v^{(j)}(r; i, n, \eta)$ via $\mathcal{H}^{i',n'}(\eta') \to \mathcal{H}^{i,n}(\eta)$.

Local-global compatibility for automorphic representations is given by the following condition, with notation as above:

Condition 4.2. (GalRep-p) *The pair (π, r) satisfies* (GalRep-p) *if there exists a unique, continuous, semisimple Galois representation*

$$R_{p,\iota}(\pi, r) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_m(\overline{\mathbb{Q}}_p)$$

such that for every $v \notin \mathrm{Ram}(\pi) \cup \{p\}$, one has

$$R_{p,\iota}(\pi, r)(\mathrm{Frob}_v)^{ss} = \mathrm{Class}_{p,\iota}(\pi_v, r_v)$$

as $\mathrm{GL}_m(\overline{\mathbb{Q}}_p)$-conjugacy classes.

Compatibility for torsion pseudo-representations is given by the following condition:

Condition 4.3. (GalPseudoRep-p) *The tuple $(r; i, n, \eta)$ satisfies* (GalPseudoRep-p) *if there exists a unique, continuous pseudo-representation*

$$R_{p,\iota}(r; i, n, \eta) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathcal{H}^{i,n}(\eta)$$

such that for every $v \notin \mathrm{Ram}(G) \cup \{p\}$, one has

$$R_{p,\iota}(r; i, n, \eta)(\mathrm{Frob}_v^j) = T_v^{(j)}(r; i, n, \eta)$$

4.2 Factorization Theorem

4.2.1 Statement of Factorization

For every triple (i, n, η) with $i \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 1}$, and $\eta \in X^*_{+,L}(T)$ (L-dominant character), let

$$F(i, n, \eta) = \left\{\eta' \in X^*_{+,L}(T) : \mathcal{H} \to \mathcal{H}^{i,n}(\eta) \text{ factors through } \mathcal{H} \to \mathcal{H}^{0,n}(\eta')\right\}$$

We want to show that $F(i, n, \eta)$ is large, which is the subject of the following theorem. This was proven in [GK19a] Theorem 8.2.1 for (not necessarily proper) Hodge type Shimura data under some additional mild hypotheses.

Theorem 4.4. *Let $\eta \in X^*_{+,L}(T)$. Assume that (G, X) is a proper abelian type Shimura datum such that $G = G^c$. Then*

(i) There exists an arithmetic progression A such that $\eta + a\,\mathrm{grif}(G, \mu) \in F(i, n, \eta)$ for all $a \in A \cap \mathbb{Z}_{\geq 1}$.

(ii) Let $\mathcal{C} \subset X^*(T)$ be the global sections cone from §2.3.4. For all $\nu \in \mathcal{C}$ and $\eta_1 \in F(i, n, \eta)$ there exists $m = m(\nu, n) \in \mathbb{Z}_{\geq 1}$ such that $\eta_1 + jm\nu \in F(i, n, \eta)$ for all $j \in \mathbb{Z}_{\geq 1}$.

(iii) For all $\delta \in \mathbb{R}_{\geq 0}$, $F(i, n, \eta)$ contains a δ-regular character (see Definition 2.17).

4.2.2 Proof of Part (i)

We first prove part (i). As in [GK19a], the proof breaks into three parts: "descent", "weight increase", and "ascent". We will make use of the following lemma ([GK19a] Lemma 8.3.1):

Lemma 4.5. *Suppose R is a $\mathbb{Z}_p$-algebra and $\epsilon : M' \to M$ is a morphism of $R\mathcal{H}$-modules. Write $\mathcal{H}_M$ (resp. $\mathcal{H}_{M'}$) for the image of $\mathcal{H}$ in $\mathrm{End}_R(M)$ (resp. $\mathrm{End}_R(M')$). If ϵ is injective (resp. surjective) then the map $\mathcal{H} \twoheadrightarrow \mathcal{H}_{M'}$ (resp. $\mathcal{H} \twoheadrightarrow \mathcal{H}_M$) factors through $\mathcal{H}_M$ (resp. $\mathcal{H}_{M'}$).*

Proof. Let $\alpha : \mathcal{H} \to \mathrm{End}_R(M)$ and $\alpha' : \mathcal{H} \to \mathrm{End}_R(M')$ denote the ring homomorphisms defining the module structures on M and M', respectively. Then $\mathcal{H}_M$ and $\mathcal{H}_{M'}$ satisfy the short exact sequences

$$0 \to \ker(\alpha) \to \mathcal{H} \to \mathcal{H}_M \to 0 \qquad \text{and} \qquad 0 \to \ker(\alpha') \to \mathcal{H} \to \mathcal{H}_{M'} \to 0$$

For each $T \in \mathcal{H}$, we have a commutative diagram

$$\begin{tikzcd} M \arrow[r, "\alpha(T)"] \arrow[d, "\epsilon"'] & M \arrow[d, "\epsilon"] \\ M' \arrow[r, "\alpha'(T)"'] & M' \end{tikzcd}$$

Suppose ϵ is injective. For any $T \in \ker(\alpha')$, the above commutative diagram gives us $\epsilon \circ \alpha(T) = \alpha'(T) \circ \epsilon = 0$, hence by injectivity of ϵ, we must have that $\alpha(T) = 0$. Thus $\ker(\alpha') \subset \ker(\alpha)$, and therefore the map $\mathcal{H} \twoheadrightarrow \mathcal{H}_{M'}$ factors through $\mathcal{H}_M$ as desired. The case when ϵ is surjective is proved similarly. □

Lemma 4.6. (Descent) *Suppose (i, n, η) is a triple as above. Then there exists $b \in \mathbb{Z}_{\geq 1}$ and a Hasse regular sequence $\mathcal{Z} = (Z_j, a_j, f_j)_{j=0}^{i}$ such that*

$$\mathcal{H} \twoheadrightarrow \mathcal{H}^{i,n}(\eta) \textit{ factors through } \mathcal{H} \twoheadrightarrow \mathcal{H}^{0,n}_{Z_i}(\eta + b\,\mathrm{grif}(G, \mu))$$

Proof. The proof follows the structure of [GK19a] Lemma 8.3.2, but with some simplifications since we are in the setting of proper Shimura varieties. To get an idea of the proof, we first handle the case when $i = 0$. Let $h_d \in H^0(\mathcal{S}^n_K, \omega^b)$ be the length Hasse invariant for the dth length stratum, where $b \geq 1$ is some integer. Since h_d is injective, multiplication by h_d gives a Hecke-equivariant injection

$$H^0(\mathcal{S}^n_K, \mathcal{V}(\eta)) \hookrightarrow H^0(\mathcal{S}^n_K, \mathcal{V}(\eta) \otimes \omega^b)$$

and hence using Lemma 4.5 gives the desired factorization. In this case, $\mathcal{Z} = (\mathcal{S}^n_K, b, h_d)$ is the Hasse regular sequence of length 0.

Now, suppose $i > 0$. We prove by induction on J that for all $0 \leq J \leq i$ there exists a Hasse regular sequence $\mathcal{Z}_J = (Z_j, a_j, f_j)_{j=0}^{J}$, an integer b_J, and a factorization

$$\mathcal{H} \twoheadrightarrow \mathcal{H}^{i,n}(\eta) \text{ factors through } \mathcal{H} \twoheadrightarrow \mathcal{H}^{i-J,n}_{Z_J}(\eta + b_J\,\mathrm{grif}(G, \mu))$$

For the base case $J = 0$, we may simply take $b_J = 0$; in subsequent steps we will have $b_J \geq 1$ as desired. For the induction step, we will show that a Hasse regular sequence $\mathcal{Z}_J$ as above can be extended by some $(Z_{J+1}, a_{J+1}, f_{J+1})$ and $b_{J+1} \geq 1$ to a Hasse regular sequence $\mathcal{Z}_{J+1}$ satisfying

$$\mathcal{H} \twoheadrightarrow \mathcal{H}^{i,n}(\eta) \text{ factors through } \mathcal{H} \twoheadrightarrow \mathcal{H}^{i-(J+1),n}_{Z_{J+1}}(\eta + b_{J+1}\,\mathrm{grif}(G,\mu))$$

By the induction hypothesis, $H^{i-J}(Z_J, \mathcal{V}(\eta) \otimes \omega^{b_J}) \neq 0$. Since the Griffiths bundle ω is ample by Proposition 2.21, there exists $m \geq 1$ such that

$$H^{i-J}(Z_J, \mathcal{V}(\eta) \otimes \omega^{b_J + m a_J}) = 0$$

Now, $f_J \in H^0(Z_J, \omega^{a_J})$ is injective and Hecke-equivariant, hence multiplication by f_J^m gives a Hecke-equivariant short exact sequence

$$0 \to \mathcal{V}(\eta) \otimes \omega^{b_J} \to \mathcal{V}(\eta) \otimes \omega^{b_J + m a_J} \to \big(\mathcal{V}(\eta) \otimes \omega^{b_J + m a_J}\big)_{Z(f_J^m)} \to 0$$

where $Z(f_J^m)$ is the zero locus of f_J^m. Let $Z_{J+1} := Z(f_J^m)$ and $b_{J+1} := b_J + m a_J$. By the cohomological vanishing above, the induced long exact sequence on cohomology yields the surjection

$$H^{i-(J+1)}(Z_{J+1}, \mathcal{V}(\eta) \otimes \omega^{b_{J+1}}) \twoheadrightarrow H^{i-J}(Z_J, \mathcal{V}(\eta) \otimes \omega^{b_J})$$

and hence Lemma 4.5 gives the desired factorization. Lastly, a_{J+1} and f_{J+1} are given by applying [GK19a] Theorem 5.1.1 to the length Hasse invariant $h_{d-(J+1)}$. □

We now fix the Hasse regular sequence $\mathcal{Z} = (Z_j, a_j, f_j)_{j=0}^{i}$ and the integer $b \in \mathbb{Z}_{\geq 1}$ afforded by the previous lemma.

Lemma 4.7. (Weight increase) *For all sufficiently large $s \in b + a_i\mathbb{Z}$, one has*

$$\mathcal{H} \twoheadrightarrow \mathcal{H}^{0,n}_{Z_i}(\eta + b\,\mathrm{grif}(G,\mu)) \text{ factors through } \mathcal{H} \twoheadrightarrow \mathcal{H}^{0,n}_{Z_i}(\eta + s\,\mathrm{grif}(G,\mu))$$

and for all $0 \leq j \leq i-1$,

$$H^1(Z_j, \mathcal{V}(\eta) \otimes \omega^{s-a_j}) = 0$$

Proof. We use a similar approach as the previous lemma, following [GK19a] Lemma 8.3.3. Consider the injective, $G(\mathbb{A}^{\infty,p})$-invariant section $f_i \in H^0(Z_i, \omega^{a_i})$ coming from $\mathcal{Z}$. For each $m \geq 1$, multiplication by f_i^m yields a Hecke-equivariant injection

$$H^0(Z_i, \mathcal{V}(\eta) \otimes \omega^{b}) \hookrightarrow H^0(Z_i, \mathcal{V}(\eta) \otimes \omega^{b + m a_i})$$

and thus Lemma 4.5 gives us the desired factorization. The vanishing is given by the ampleness of the Griffiths bundle, Proposition 2.21, for $m \gg 0$. □

Lemma 4.8. (Ascent) *Suppose s satisfies the properties of Lemma 4.7. Then*

$$\mathcal{H} \to \mathcal{H}^{0,n}_{Z_i}(\eta + s\,\mathrm{grif}(G,\mu)) \text{ factors through } \mathcal{H} \to \mathcal{H}^{0,n}(\eta + s\,\mathrm{grif}(G,\mu))$$

Proof. We follow [GK19a] Lemma 8.3.4. We will prove by downward induction on J that

$$\mathcal{H} \to \mathcal{H}^{0,n}_{Z_i}(\eta + s\,\mathrm{grif}(G,\mu)) \text{ factors through } \mathcal{H} \to \mathcal{H}^{0,n}_{Z_J}(\eta + s\,\mathrm{grif}(G,\mu))$$

for all $0 \leq J \leq i$. The base case $J = i$ is trivial. So, assuming the above factorization holds, we show that

$$\mathcal{H} \to \mathcal{H}^{0,n}_{Z_J}(\eta + s\,\mathrm{grif}(G,\mu)) \text{ factors through } \mathcal{H} \to \mathcal{H}^{0,n}_{Z_{J-1}}(\eta + s\,\mathrm{grif}(G,\mu))$$

which combined with the above factorization yields the desired factorization.

We employ a similar method as the previous two lemmas. Multiplication by the section $f_{J-1} \in H^0(Z_{J-1}, \omega^{a_{J-1}})$ gives a short exact sequence of sheaves

$$0 \to \mathcal{V}(\eta) \otimes \omega^{s-a_{J-1}} \to \mathcal{V}(\eta) \otimes \omega^s \to (\mathcal{V}(\eta) \otimes \omega^s)_{Z_J} \to 0$$

The induced long exact sequence on cohomology yields

$$H^0(Z_{J-1}, \mathcal{V}(\eta) \otimes \omega^s) \to H^0(Z_J, \mathcal{V}(\eta) \otimes \omega^s) \to H^1(Z_{J-1}, \mathcal{V}(\eta) \otimes \omega^{s-a_{J-1}})$$

and by Lemma 4.7 the rightmost term is 0. Thus, Lemma 4.5 gives us the desired factorization. □

Proof of Theorem 4.4(i). Combine Lemmas 4.6, 4.7, and 4.8. □

4.2.3 Proof of Parts (ii) and (iii)

We now prove parts (ii) and (iii) of Theorem 4.4. To do this, we need to introduce the flag space associated to the Shimura variety $\mathcal{S}_K$. Recall from §3.2.1 that the zip period map $\zeta_K : \mathcal{S}_K \to G\text{-}\mathbf{Zip}^\mu$ is equivalent to giving a universal G-zip (I_G, I_P, I_Q, ι) on $\mathcal{S}_K$. The quotient of I_P by the Borel subgroup B is represented by a smooth, projective $\mathcal{S}_K$-scheme denoted by $\mathcal{F}_K$, the *flag space* of $\mathcal{S}_K$.

In §2.3.4 we defined the stack $G\text{-}\mathbf{ZipFlag}^\mu$ of G-zip flags of type μ, which comes equipped with a natural projection $\pi : G\text{-}\mathbf{ZipFlag}^\mu \to G\text{-}\mathbf{Zip}^\mu$. The flag space $\mathcal{F}_K$ is naturally the pullback of the morphisms ζ_K and π:

$$\begin{tikzcd} \mathcal{F}_K \arrow[r, "\tilde{\pi}"] \arrow[d, "\tilde{\zeta}_K"'] & \mathcal{S}_K \arrow[d, "\zeta_K"] \\ G\text{-}\mathbf{ZipFlag}^\mu \arrow[r, "\pi"] & G\text{-}\mathbf{Zip}^\mu \end{tikzcd}$$

The tower $(\mathcal{F}_K)_{K^p}$ admits an action by $G(\mathbb{A}^{\infty,p})$, and the system of maps $\tilde{\pi} : \mathcal{F}_K \to \mathcal{S}_K$ is $G(\mathbb{A}^{\infty,p})$-equivariant. As in §3.1.3, for any character $\eta \in X^*(T)$ we can define a line bundle $\mathcal{L}(\eta)$ on $\mathcal{F}_K$. It is compatible with the above diagram in the sense that

$$\mathcal{L}(\eta) = \tilde{\pi}^*\mathcal{V}(\eta) \qquad \text{and} \qquad \mathcal{L}(\eta) = \tilde{\zeta}_K^*\mathscr{L}_{\mathrm{Flag}}(\eta)$$

Moreover, there is a canonical $G(\mathbb{A}^{\infty,p})$-equivariant identification

$$\tilde{\pi}_*\mathcal{L}(\eta) = \mathcal{V}(\eta)$$

Note that the pushforward makes sense since the k-fibers of $\tilde{\pi}$ are isomorphic to $B/(B \cap L)$, hence the space of global sections $H^0(B/(B \cap L), \mathcal{L}(\eta))$ has constant dimension across the fibers. Furthermore, if $\eta \notin X^*_{+,L}(T)$, we see that $H^0(B/(B\cap L), \mathcal{L}(\eta)) = 0$, hence $\tilde{\pi}_*\mathcal{L}(\eta) = \mathcal{V}(\eta) = 0$. Thus, if $\mathcal{L}(\eta)$ admits a nonzero global section, we must have $\eta \in X^*_{+,L}(T)$. In particular, since every $\eta \in \mathcal{C}$ gives a nonzero global section of $\mathscr{L}_{\mathrm{Flag}}(\eta)$ and $\mathcal{L}(\eta) = \tilde{\zeta}_K^* \mathscr{L}_{\mathrm{Flag}}(\eta)$, we see that $\mathcal{C} \subset X^*_{+,L}(T)$.

A main motivation for considering the flag space is the following comparison of cohomologies. Write $\mathcal{F}^n_K$ for the reduction mod p^n of $\mathcal{F}_K$ and $\tilde{\pi} : \mathcal{F}^n_K \to \mathcal{S}^n_K$ for the natural projection. Let $\eta \in X^*_{+,L}(T)$. Then by [GK19a] Remark 9.1.3, for all $i \in \mathbb{Z}_{\geq 0}$ there is a Hecke-equivariant isomorphism

$$H^i(\mathcal{F}^n_K, \mathcal{L}(\eta)) \cong H^i(\mathcal{S}^n_K, \mathcal{V}(\eta))$$

We can now prove the rest of Theorem 4.4.

Proof of Theorem 4.4(ii). The proof is essentially unchanged from the proof in [GK19a] §9.2. We give the basic idea here. Using Proposition 2.23 along with [GK19a] Theorem 5.1.1, we obtain an injective, $G(\mathbb{A}^{\infty,p})$-equivariant section $h_\nu \in H^0(\mathcal{F}^n_K, \mathcal{L}(m\nu))$ for some $m \geq 1$. For all $j \geq 1$, multiplication by h_ν^j induces a Hecke-equivariant injection

$$H^0(\mathcal{F}^n_K, \mathcal{L}(\eta_1)) \hookrightarrow H^0(\mathcal{F}^n_K, \mathcal{L}(\eta_1 + jm\nu))$$

Applying the comparison of cohomologies given above, we thus obtain a Hecke-equivariant injection

$$H^0(\mathcal{S}^n_K, \mathcal{V}(\eta_1)) \hookrightarrow H^0(\mathcal{S}^n_K, \mathcal{V}(\eta_1 + jm\nu))$$

Assuming that $\eta_1 \in F(i, n, \eta)$, an application of Lemma 4.5 gives us $\eta_1 + jm\nu \in F(i, n, \eta)$. □

Proof of Theorem 4.4(iii). Given Theorem 4.4(i) and Theorem 4.4(ii), the proof is given exactly as in [GK19a] §9.2. □

4.3 Galois Representations

Given Theorem 4.4, we can now state the results on Galois representations. For the remainder of this section, let (G, X) denote an abelian type Shimura datum such that $G^c = G$ and G^{der} is $\mathbb{Q}$-anisotropic, so that the resulting Shimura variety is proper. Fix a prime $p \notin \mathrm{Ram}(G) \cup \{2\}$ and an integral model $\mathscr{S}_K$ over $\mathcal{O}_{E,\mathfrak{p}}$ as in §3.1.2. Recall the notion of δ-regular character from Definition 2.17.

4.3.1 Galois Pseudo-representations Associated to Torsion Hecke Coefficients

The statement for associating pseudo-representations to systems of torsion Hecke coefficients is analogous [GK19a] Theorem 10.4.1, with the only difference being that we state our theorem for proper abelian type Shimura data rather than for general Hodge type. Thus, we can drop qualifiers such as "cuspidal" since every automorphic representation is cuspidal in this case.

Theorem 4.9. *Let (G,X) be a proper abelian type Shimura datum with $G^c = G$. Let $\delta \in \mathbb{R}_{\geq 0}$ and $r : {}^L G \to \mathrm{GL}_m$ be a representation. Suppose that, for every δ-regular C-algebraic automorphic representation π with π_∞ discrete series, the pair (π, r) satisfies satisfies Condition 4.2 (GalRep-p). Then for every triple (i,n,η) with $i \geq 0$ and $n \geq 1$, the tuple $(r; i, n, \eta)$ satisfies Condition 4.3 (GalPseudoRep-p).*

Proof. Fix a triple (i,n,η) and $\delta \in \mathbb{R}_{\geq 0}$. For technical reasons (see below), we set

$$\delta' = \delta + 1 + \max\{|\langle \rho, \alpha^\vee \rangle| : \alpha \in \Phi(G,T)\}$$

By Theorem 4.4 (iii) there exists $\nu \in F(i,n,\eta)$ which is δ'-regular. Since $\mathrm{grif}(G,\mu) \in \mathcal{C}$ and $\nu \in F(i,n,\eta)$, Theorem 4.4 (ii) gives us an integer $k \geq 1$ such that $\nu + ak\,\mathrm{grif}(G,\mu) \in F(i,n,\eta)$ for all integers $a \geq 1$. Fix such a k, and fix a sufficiently large integer $a \geq 1$ such that

(i) $\nu + ak\,\mathrm{grif}(G,\mu)$ is δ'-regular

(ii) the map

$$H^0(\mathscr{S}_K, \mathcal{V}(\nu + ak\,\mathrm{grif}(G,\mu))) \to H^0(\mathcal{S}_K^n, \mathcal{V}(\nu + ak\,\mathrm{grif}(G,\mu))) \tag{4.2}$$

is surjective (this can be assured by the ampleness of the Griffiths bundle, Proposition 2.21)

For every Hecke eigenform $f \in H^0(\mathscr{S}_K, \mathcal{V}(\nu + ak\,\mathrm{grif}(G,\mu)))$, the choice of δ' guarantees that the automorphic representation $\pi(f)$ has a discrete series Archimedean component, hence it satisfies the hypotheses in the theorem statement concerning (GalRep-p). Thus, applying this condition across all Hecke eigenforms in $H^0(\mathscr{S}_K, \mathcal{V}(\nu + ak\,\mathrm{grif}(G,\mu)))$ yields a unique, continuous, semisimple Galois representation

$$\rho_a : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_m(\mathcal{H}^0(\nu + ak\,\mathrm{grif}(G,\mu)) \otimes \overline{\mathbb{Q}}_p)$$

and the trace of this representation gives a pseudo-representation

$$\rho_a^{\mathrm{pseudo}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathcal{H}^0(\nu + ak\,\mathrm{grif}(G,\mu))$$

such that $\rho_a^{\mathrm{pseudo}}(\mathrm{Frob}_v^j) = T_v^{(j)}(r; 0, +, \nu + ak\,\mathrm{grif}(G,\mu))$, where the " $+$ " indicates that we have not reduced mod p^n.

Now, we obtain a factorization of $\mathcal{H} \to \mathcal{H}^{i,n}(\eta)$

$$\mathcal{H} \to \mathcal{H}^0(\nu + ak\,\mathrm{grif}(G,\mu)) \to \mathcal{H}^{0,n}(\nu + ak\,\mathrm{grif}(G,\mu)) \to \mathcal{H}^{i,n}(\eta)$$

where

(i) $\mathcal{H} \to \mathcal{H}^{0,n}(\nu + ak\,\mathrm{grif}(G,\mu))$ factors through $\mathcal{H} \to \mathcal{H}^0(\nu + ak\,\mathrm{grif}(G,\mu))$ by the surjectivity of (4.2)

(ii) $\mathcal{H} \to \mathcal{H}^{i,n}(\eta)$ factors through $\mathcal{H} \to \mathcal{H}^{0,n}(\nu + ak\,\mathrm{grif}(G,\mu))$ since $\nu + ak\,\mathrm{grif}(G,\mu) \in F(i,n,\eta)$

Thus, composing the map $\mathcal{H}^{0,n}(\nu + ak\,\mathrm{grif}(G,\mu)) \to \mathcal{H}^{i,n}(\eta)$ with ρ_a^{pseudo} gives the desired torsion pseudo-representation

$$R_{p,\iota}(r; i, n, \eta) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathcal{H}^{i,n}(\eta)$$

which satisfies Condition 4.3 (GalPseudoRep-p). $\square$

4.3.2 Galois Representations Associated to Non-degenrate LDS

Given Theorem 4.9, we can state and prove the following theorem, which is given as Theorem 10.5.1 in [GK19a] in the general Hodge type case.

Theorem 4.10. *Let (G, X) be a proper abelian type Shimura datum with $G^c = G$. Suppose there exists $\delta \in \mathbb{R}_{\geq 0}$ and $r : {}^L G \to \mathrm{GL}_m$ such that for every δ-regular C-algebraic automorphic representation π' with π'_∞ discrete series, the pair (π', r) satisfies Condition 4.2 (GalRep-p). Let π be an automorphic representation of G with π_∞ a C-algebraic, non-degenerate LDS. Then the pair (π, r) satisfies Condition 4.2 (GalRep-p).*

Proof. Let π be as in the theorem statement. Then π gives rise to a character $\eta \in X^*_{+,L}(T)$ and a Hecke eigenclass γ_π in $H^i(\mathscr{S}_K, \mathcal{V}(\eta))$, where the degree i is determined by π_∞. Write

$$\theta : \mathcal{H}^i(\eta) \to \mathcal{O}_{E,\mathfrak{p}}$$

for the eigenvalue map of γ_π and

$$\theta_n : \mathcal{H}^{i,n}(\eta) \to \mathcal{O}_{E,\mathfrak{p}}/\mathfrak{p}^n$$

for its reduction mod $\mathfrak{p}^n$. Then

$$\theta(T_v^{(j)}(r; i, +, \eta)) = \mathrm{tr}(\mathrm{Class}_{p,\iota}(\pi_v, r_v)^j)$$

by the bijection (4.1). Now, we apply Theorem 4.9, we obtain a continuous torsion pseudo-representation $R_{p,\iota}(r; i, n, \eta)$ satisfying Condition 4.3. Since $\theta_n(T_v^{(j)}(r; i, n, \eta))$ is the reduction mod $\mathfrak{p}^n$ of $\theta(T_v^{(j)}(r; i, +, \eta))$, the maps $\theta_n \circ R_{p,\iota}(r; i, n, \eta)$ form a $\mathfrak{p}$-adic system of pseudo-representations such that

$$\theta_n \circ R_{p,\iota}(r; i, n, \eta)(\mathrm{Frob}^j) \equiv \mathrm{tr}(\mathrm{Class}_{p,\iota}(\pi_v, r_v)^j) \mod \mathfrak{p}^n$$

The limit of these is a pseudo-representation $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathcal{O}_{E,\mathfrak{p}} \hookrightarrow \overline{\mathbb{Q}}_p$ such that

$$\chi(\mathrm{Frob}^j) = \mathrm{tr}(\mathrm{Class}_{p,\iota}(\pi_v, r_v)^j)$$

for all $j \geq 1$. By Theorem 1 in [Tay91], χ is the trace of a representation $R_{p,\iota}(\pi, r)$ satisfying Condition 4.2. □

5 Beyond the Proper Abelian Type Case

In this section, we discuss the current status of extending this work to the non-proper abelian type case, as well as the exceptional case. At this time, not all of the necessary results are known; we highlight what work must be done in order to extend these results.

5.1 Non-proper Abelian Type Shimura Varieties

Recent results on compactifications of non-proper abelian type Shimura varieties give a pathway to extending the results of this paper, but not all the pieces are in place yet. We discuss what needs to be done below.

The results shown in this paper were proven in the non-proper Hodge type case by Goldring and Koskivirta in [GK19a]. They made use of the following constructions:

(i) smooth integral models of toroidal compactifications of Hodge type Shimura varieties, as constructed by Madapusi Pera in [MP19];

(ii) integral models of (sub-)canonical extensions of automorphic vector bundles, also given by Madapusi Pera ([MP19])

(iii) an extension of the EO-stratification map ζ to the mod p reduction of the toroidal compactification, which is Theorem 6.2.1 in [GK19a] (also independently proven by Shengkai Mao in [Mao25])

Using the above ingredients, Goldring-Koskivirta were able to associate Galois representations to automorphic representations when G comes from a general (not necessarily proper) Hodge type Shimura datum.

Some progress has been made in establishing these results for the abelian type case. The recent preprint by Peihang Wu ([Wu25]) constructs integral models of toroidal compactifications of abelian type Shimura varieties, which covers item (i). In the paper, Wu states that he and Shengkai Mao are currently working on the problem of constructing (sub-)canonical extensions (item (ii)), but as of writing this has not been accomplished yet. Lastly, we believe that both Wu and Mao are also tackling item (iii).

With these results on the horizon, one should be able to fully extend the results of [GK19a] to the abelian type case, under some potential mild hypotheses similar to Conditions 6.4.2 and 7.1.2 in [GK19a].

5.2 Exceptional Shimura Varieties

Aside from compactifications of abelain type Shimura varieties, we can also attempt to generalize the results of this paper to exceptional (proper) Shimura varieties. The recent preprint of Bakker-Shankar-Tsimerman ([BST24]) provides integral models of exceptional Shimura varieties for sufficiently large primes. As stated in the introduction, in order to construct strata Hasse invariants on a general Shimura variety, one needs

1. an EO-stratification map ζ

2. an automorphic vector bundle which admits group-theoretic Hasse invariants

One could feasibly use the Griffiths bundle for exceptional Shimura varieties, so we really just need to provide the EO-stratification map ζ. It is not clear to the author how this would be done: the Hodge type case makes use of the universal abelian scheme, and the abelian type case makes use of the Hodge type case via the association described in §3.2.2. It is possible that the results of [BST24] can provide one with a crystalline canonical model as in

[Lov17a], from which one could produce the EO-stratification map as in [SZ22], but this is not clear.

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